Available online at https://ganitjournal.bdmathsociety.org/

GANIT

GANIT: Journal of Bangladesh Mathematical Society

**GANIT** *J. Bangladesh Math. Soc.* 44.2 (2024) 028–038
DOI: https://doi.org/10.3329/ganit.v44i2.78530

# Correlation Coefficients of Complex Fuzzy Sets and Their Application

Md. Yasin Ali[a] *, Abedul Hadi[a]

[a]*Faculty of Science and Engineering, University of Information Technology & Sciences, Dhaka-1212, Bangladesh*

## ABSTRACT

Complex fuzzy set (CFS) is an extension of the fuzzy set (FS) which can deal with ambiguity by allowing a complex-valued membership degree of an element of a universal set. A host of researchers studied the complex fuzzy sets in theoretical and practical due to their amplitude term and phase term membership degrees. At present, various applications of fuzzy correlation and correlation coefficients have emerged by numerous researchers. But most of the works are related to real fuzzy data. In this article, we introduce the concept of the correlation coefficient of the complex fuzzy sets and some of its related properties are described. Furthermore, an application of the correlation coefficient of the complex fuzzy sets in pattern recognition is illustrated. To show the reliability and validity of our technique, we explain a comparative study with the existing method.



**Received:** February 28, 2024 **Accepted:** July 27, 2024 **Published Online:** December 30, 2024

**Keywords:** Fuzzy set; Complex fuzzy set; Correlation coefficient of complex fuzzy sets; Pattern recognition.

***AMS Subject Classifications 2024:*** 03E72, 03E75, 62E86.

## 1. Introduction

Many aspect of our real life, we face some problems where the data sets are imprecise. To describe such situation Zadeh [1] introduced Fuzzy Set (FS) theory which was an appropriate tool to describe imprecise data. Later a lot of generalization of FS has been done by a host of researchers and many applications of it have arisen over the years. One of the generalizations of FS is complex fuzzy set (CFS), proposed Ramot et al. [2] is also capable of dealing with ambiguity by allowing amplitude term and phase term membership degrees.

On the other hand, Correlation coefficients are used to measure the strength and direction of the linear relationships between pairs of variables. When both variables are normally ambiguous, we use fuzzy correlation coefficient. In the FS theory, the concept of correlation coefficient was first introduced Gerstenkorn and Manko [3] in 1991. They defined the correlation coefficient for two intuitionistic fuzzy sets. Later a group of researchers studied the correlation coefficients for intuitionistic fuzzy sets in different trends and in other spaces such as [4-9]. In 1999, the correlation coefficient for fuzzy set was studied by Chiang and Lin [10]. The correlation coefficients for the FSs are also studied [11-14]. In the existing studies of correlation coefficient in FSs and their extension, the data are handled with the help of degree of membership which range are considered in the interval $[0,1]$. But sometimes this may be insufficient and consequently, it may be affected our desire result. An alternative to these, CFS that can handle any data which range of membership degrees may be extended from real interval $[0,1]$ to the unit disc of a complex plane. Thus, the CFS can handle the two-dimensional information in a single set. By motivating this, we develop a correlation coefficient of CFSs where the membership degree represents two-dimensional information for a single set. We also describe various properties of its. Finally, an application of the correlation coefficient of CFSs in pattern recognition is established.

The article is organized as follows: In section 2, we describe some definitions which are essential to rest of the paper. In section 3, Correlation coefficients of complex fuzzy sets are discussed with some its properties. In section 4, an application is illustrated by using our proposed methods. In section 5, comparative studies are also discussed with some existing methods. Finally, concluding remarks are given.

## 2. Preliminaries

**Definition 1**. [2] A complex fuzzy set (CFS), defined on a universal set $X$ is characterized by a membership function $\mu_A(x)$ that assigns a complex-valued grade of membership in $A$ to any element $x \in X$ . By definition, all values of $\mu_A(x)$ lie within the unit circle in the complex plane, and are expressed of the form $r_{A(x)}. e^{i\omega_A(x)}$, where $i = \sqrt{-1}$, $r_{A(x)}$ and $\omega_A(x)$ are both real-valued, and $r_A(x) \in [0, 1]$ and $\omega_A(x) \in [0, 2\pi]$. The complex fuzzy set may be represented as the set of ordered pairs

$$A = \{(x, \mu_A(x)) : x \in X\} = \{(x, r_A(x). e^{i\omega_A(x)}) : x \in X\}.$$

Where $'.'$ denotes algebraic product.

The complement of $A$ is denoted as $A'$ and defined by

$$A' = (x, \mu'_A(x) : x \in X),$$

where, $\mu'_A(x) = r'_A(x). e^{i\omega'_A(x)}$ in which $r'_A(x). e^{i\omega'_A(x)} = \left(1 - r_A(x). e^{i(2\pi - \omega_A(x))}\right)$.

## 3. Correlation coefficient of complex fuzzy sets

In this section, we propose correlation coefficients between two CFSs which are very important in various engineering problems where the amplitude term and phase term are used simultaneously.

**Definition 2.** Let $A = \{(x, r_A(x). e^{i\omega_A(x)}) : x \in X\}$ and $B = \{(x, r_B(x). e^{i\omega_B(x)}) : x \in X\}$ be two CFSs on $X$. Then the correlation coefficient between $A$ and $B$ is denoted by $K(A, B)$, is defined as

$$K(A, B) = \frac{C(A,B)}{\sqrt{E(A).E(B)}} \tag{1}$$

where,

$$C(A, B) = \frac{1}{2n}\left[\sum_{j=1}^{n}\left\{\left(r_A(x_j)r_B(x_j) + r'_A(x_j)\mu'_B(x_j)\right) + \frac{1}{4\pi^2}\left(\omega_A(x_j)\omega_B(x_j) + \omega'_A(x_j)\omega'_B(x_j)\right)\right\}\right] \tag{2}$$

$$E(A) = \frac{1}{2n}\left[\sum_{j=1}^{n}\left\{\left(r_A^2(x_j) + r_A'^2(x_j)\right) + \frac{1}{4\pi^2}\left(\omega_A^2(x_j) + \omega_A'^2(x_j)\right)\right\}\right] \tag{3}$$

$$E(B) = \frac{1}{2n}\left[\sum_{j=1}^{n}\left\{\left(r_B^2(x_j) + r_B'^2(x_j)\right) + \frac{1}{4\pi^2}\left(\omega_B^2(x_j) + \omega_B'^2(x_j)\right)\right\}\right] \tag{4}$$

**Example 1.** Let $U = \{x_1, x_2, x_3, x_4\}$ be the universal set and

$A = \{(x_1, 0.8e^{i\,2\pi(0.5)}), (x_2, 0.7e^{i\,2\pi(0.4)}), (x_3, 0.6e^{i\,2\pi(0.3)}), (x_4, 0.9e^{i\,2\pi(0.8)})\}$ and

$B = \{(x_1, 0.4e^{i\,2\pi(0.5)}), (x_2, 0.8e^{i\,2\pi(0.6)}), (x_3, 0.5e^{i\,2\pi(0.4)}), (x_4, 0.3e^{i\,2\pi(0.2)})\}$ be two

CFSs on $X$.

Then we have,

$A' = \left\{(x_1, 0.2e^{i\,2\pi(0.5)}), (x_2, 0.3e^{i\,2\pi(0.6)}), (x_3, 0.4e^{i\,2\pi(0.7)})', (x_4, 0.1e^{i\,2\pi(0.2)})\right\}$ and

$B' = \{(x_1, 0.6e^{i\,2\pi(0.5)}), (x_2, 0.2e^{i\,2\pi(0.4)}), (x_3, 0.5e^{i\,2\pi(0.6)}), (x_4, 0.7e^{i\,2\pi(0.8)})\}$.

Now by equations (2), (3) and (4) we have

$$C(A,B)=\frac{1}{2n}\left[\sum_{j=1}^{n}\left\{\left(r_A(x_j)r_B(x_j)+r_A'(x_j)\mu_B'(x_j)\right)+\frac{1}{4\pi^2}\left(\omega_A(x_j)\omega_B(x_j)+\omega_A'(x_j)\omega_B'(x_j)\right)\right\}\right]$$

$$=\frac{1}{2\times 4}\Big[(0.8\times 0.4+0.2\times 0.6)+\frac{1}{4\pi^2}\left(2\pi(0.5)\times 2\pi(0.5)+2\pi(0.5)\times 2\pi(0.5)\right)+(0.7\times 0.8+0.3\times\ 0.2)+\frac{1}{4\pi^2}\left(2\pi(0.4)\times 2\pi(0.6)+2\pi(0.6)\times 2\pi(0.4)\right)+(0.6\times 0.5+0.4\times 0.5)+\frac{1}{4\pi^2}\left(2\pi(0.3)\times 2\pi(0.4)+2\pi(0.7)\times 2\pi(0.6)\right)+(0.9\times 0.3+0.1\times 0.7)+\frac{1}{4\pi^2}\left(2\pi(0.8)\times 2\pi(0.2)+2\pi(0.2)\times 2\pi(0.8)\right)\Big]$$

$$=0.4675$$

$$E(A)=\frac{1}{2n}\left[\sum_{j=1}^{n}\left\{\left(r_A^2(x_j)+r_A'^2(x_j)\right)+\frac{1}{4\pi^2}\left(\omega_A^2(x_j)+\omega_A'^2(x_j)\right)\right\}\right]$$

$$=\frac{1}{2\times 4}\Big[(0.8)^2+(0.2)^2+\frac{1}{4\pi^2}((2\pi\times 0.5)^2+(2\pi\times 0.5)^2)+(0.7)^2+(0.3)^2+\frac{1}{4\pi^2}((2\pi\times 0.4)^2+(2\pi\times 0.6)^2)+(0.6)^2+(0.4)^2+\frac{1}{4\pi^2}((2\pi\times 0.3)^2+(2\pi\times 0.7)^2)+(0.3)^2+(0.7)^2+\frac{1}{4\pi^2}((2\pi\times 0.8)^2+(2\pi\times 0.2)^2)\Big]$$

$$=0.58$$

$$E(B)=\frac{1}{2n}\left[\sum_{j=1}^{n}\left\{\left(r_B^2(x_j)+r_B'^2(x_j)\right)+\frac{1}{4\pi^2}\left(\omega_B^2(x_j)+\omega_B'^2(x_j)\right)\right\}\right]$$

$$=\frac{1}{2\times 4}\Big[(0.4)^2+(0.6)^2+\frac{1}{4\pi^2}((2\pi\times 0.5)^2+(2\pi\times 0.5)^2)+(0.8)^2+(0.2)^2+\frac{1}{4\pi^2}((2\pi\times 0.6)^2+(2\pi\times 0.4)^2)+(0.5)^2+(0.5)^2+\frac{1}{4\pi^2}((2\pi\times 0.4)^2+(2\pi\times 0.6)^2)+(0.9)^2+(0.1)^2+\frac{1}{4\pi^2}((2\pi\times 0.2)^2+(2\pi\times 0.8)^2)\Big]$$

$$=0.5925$$

Hence from the equation (1), we have

$$K(A,B)=\frac{C(A,B)}{\sqrt{E(A).E(B)}}$$

$$=\frac{0.4675}{\sqrt{0.58\times 0.5925}}$$

$$=0.797.$$

**Proposition 1.** Let $A$ and $B$ be two CFSs on $X$. Then,

1. $0\leq C(A,B)\leq 1$,
2. $C(A,B)=C(B,A)$,
3. $C(A,A)=E(A)$.

**Proof**. Trivial.

**Theorem 1.** Let $A$ and $B$ be two CFSs on $X$. Then the correlation coefficient $K(A,B)$satisfies the following properties:

1. $0\leq K(A,B)\leq 1$,
2. $K(A,B)=K(B,A)$,
3. If $A=B$, then $K(A,B)=1$.

**Proof.** Let $A=\{(x,r_A(x).e^{i\omega_A(x)}):x\in X\}$ and $B=\{(x,r_B(x).e^{i\omega_B(x)}):x\in X\}$ be two CFSs on $X$. Then we have,

1. Since $C(A,B)\geq 0$, we need to only show that, $C(A,B)\leq\sqrt{E(A).E(B)}$.

Now from the definition 2 we have,

$$C(A,B)=\frac{1}{2n}\left[\sum_{j=1}^{n}\left\{\left(r_A(x_j)r_B(x_j)+r_A'(x_j)r_B'(x_j)\right)+\frac{1}{4\pi^2}\left(\omega_A(x_j)\omega_B(x_j)+\omega_A'(x_j)\omega_B'(x_j)\right)\right\}\right]$$

$$= \frac{1}{2n}\left[\sum_{j=1}^{n}\{r_A(x_j)r_B(x_j) + r'_A(x_j)r'_B(x_j)\} + \sum_{j=1}^{n}\left\{\left(\frac{\omega_A(x_j)}{2\pi}\right)\left(\frac{\omega_B(x_j)}{2\pi}\right) + \left(\frac{\omega'_A(x_j)}{2\pi}\right)\left(\frac{\omega'_B(x_j)}{2\pi}\right)\right\}\right]$$

$$= \frac{1}{2n}[C_1(A,B) + C_2(A,B)] \quad \text{[Say].}$$

And

$$E(A) = \frac{1}{2n}\left[\sum_{j=1}^{n}\left(r_A^2(x_j) + {r'_A}^2(x_j)\right) + \sum_{j=1}^{n}\left\{\left(\frac{\omega_A(x_j)}{2\pi}\right)^2 + \left(\frac{\omega'_A(x_j)}{2\pi}\right)^2\right\}\right]$$

$$= \frac{1}{2n}[E_1(A) + E_2(A)] \quad \text{[Say],}$$

$$E(B) = \frac{1}{2n}\left[\sum_{j=1}^{n}\left(r_B^2(x_j) + {r'_B}^2(x_j)\right) + \sum_{j=1}^{n}\left\{\left(\frac{\omega_B(x_j)}{2\pi}\right)^2 + \left(\frac{\omega'_B(x_j)}{2\pi}\right)^2\right\}\right]$$

$$= \frac{1}{2n}[E_1(B) + E_2(B)] \quad \text{[Say].}$$

Now, in order to prove $C(A,B) \le \sqrt{E(A).E(B)}$, it is sufficient to prove

$C_1(A,B) \le \sqrt{E_1(A).E_1(B)}$ and $C_2(A,B) \le \sqrt{E_2(A).E_2(B)}$.

First from the Schwarz inequality we have,

$$\left(\sum_{j=1}^{n}\left(r_A^2(x_j) + {r'_A}^2(x_j)\right)\sum_{j=1}^{n}\left(r_B^2(x_j) + {r'_B}^2(x_j)\right)\right)^{\frac{1}{2}} \ge \sum_{j=1}^{n}\left[\left(r_A^2(x_j) + {r'_A}^2(x_j)\right)^{1/2}\left(r_B^2(x_j) + {r'_B}^2(x_j)\right)^{1/2}\right]$$

$$= \sum_{j=1}^{n}\begin{pmatrix} r_A^2(x_j)r_B^2(x_j) + {r'_A}^2(x_j){r'_B}^2(x_j) \\ + r_A^2(x_j){r'_B}^2(x_j) + {r'_A}^2(x_j)r_B^2(x_j)\end{pmatrix}^{1/2}.$$

Second from,

$$r_A^2(x_j){r'_B}^2(x_j) + {r'_A}^2(x_j)r_B^2(x_j) \ge 2r_A(x_j)r'_A(x_j)r'_B(x_j)r_B(x_j).$$

We can get

$$r_A^2(x_j)r_B^2(x_j) + {r'_A}^2(x_j){r'_B}^2(x_j) + r_A^2(x_j){r'_B}^2(x_j) + {r'_A}^2(x_j)r_B^2(x_j) \ge \left(r_A(x_j)r_B(x_j) + r'_A(x_j)r'_B(x_j)\right)^2.$$

Then it follows that

$$\sum_{j=1}^{n}\left(r_A^2(x_j)r_B^2(x_j) + {r'_A}^2(x_j){r'_B}^2(x_j) + r_A^2(x_j){r'_B}^2(x_j) + {r'_A}^2(x_j)r_B^2(x_j)\right)^{1/2} \ge \sum_{j=1}^{n}\left(r_A(x_j)r_B(x_j) + r'_A(x_j)r'_B(x_j)\right).$$

Hence

$$\left(\sum_{j=1}^{n}\left(r_A^2(x_j) + {r'_A}^2(x_j)\right)\sum_{j=1}^{n}\left(r_B^2(x_j) + {r'_B}^2(x_j)\right)\right)^{\frac{1}{2}} \ge \sum_{j=1}^{n}\left(r_A(x_j)r_B(x_j) + r'_A(x_j)r'_B(x_j)\right)$$

$$\ge C_1(A,B).$$

Thus

$C_1(A,B) \le \sqrt{E_1(A).E_1(B)}$.

On the other hand, first from the Schwarz inequality we have,

$$\begin{pmatrix} \sum_{j=1}^{n}\left\{\left(\frac{\omega_A(x_j)}{2\pi}\right)^2 + \left(\frac{\omega'_A(x_j)}{2\pi}\right)^2\right\} \\ \sum_{j=1}^{n}\left\{\left(\frac{\omega_B(x_j)}{2\pi}\right)^2 + \left(\frac{\omega'_B(x_j)}{2\pi}\right)^2\right\}\end{pmatrix}^{1/2} \ge \sum_{j=1}^{n}\left[\begin{matrix}\left(\left(\frac{\omega_A(x_j)}{2\pi}\right)^2 + \left(\frac{\omega'_A(x_j)}{2\pi}\right)^2\right)^{\frac{1}{2}} \\ \left(\left(\frac{\omega_B(x_j)}{2\pi}\right)^2 + \left(\frac{\omega'_B(x_j)}{2\pi}\right)^2\right)^{1/2}\end{matrix}\right]$$

$$= \sum_{j=1}^{n}\left(\left(\frac{\omega_A(x_j)}{2\pi}\right)\left(\frac{\omega_B(x_j)}{2\pi}\right)^2 + \left(\frac{\omega'_A(x_j)}{2\pi}\right)^2\left(\frac{\omega'_B(x_j)}{2\pi}\right)^2 + \left(\frac{\omega_A(x_j)}{2\pi}\right)^2\left(\frac{\omega'_B(x_j)}{2\pi}\right)^2 + \left(\frac{\omega'_A(x_j)}{2\pi}\right)^2\left(\frac{\omega_B(x_j)}{2\pi}\right)^2\right)^{1/2}.$$

Second from,

$$\left(\frac{\omega_A(x_j)}{2\pi}\right)^2\left(\frac{\omega'_B(x_j)}{2\pi}\right)^2 + \left(\frac{\omega'_A(x_j)}{2\pi}\right)^2\left(\frac{\omega_B(x_j)}{2\pi}\right)^2 \geq\ 2\frac{\omega_A(x_j)}{2\pi}\frac{\omega'_A(x_j)}{2\pi}\frac{\omega'_B(x_j)}{2\pi}\frac{\omega_B(x_j)}{2\pi},$$

we can get

$$\left(\left(\frac{\omega_A(x_j)}{2\pi}\right)^2\left(\frac{\omega_B(x_j)}{2\pi}\right)^2 + \left(\frac{\omega'_A(x_j)}{2\pi}\right)^2\left(\frac{\omega'_B(x_j)}{2\pi}\right)^2 + \left(\frac{\omega_A(x_j)}{2\pi}\right)^2\left(\frac{\omega'_B(x_j)}{2\pi}\right)^2 + \left(\frac{\omega'_A(x_j)}{2\pi}\right)^2\left(\frac{\omega_B(x_j)}{2\pi}\right)^2\right)$$

$$\geq\ \left(\frac{\omega_A(x_j)}{2\pi}\frac{\omega_B(x_j)}{2\pi} + \frac{\omega'_A(x_j)}{2\pi}\frac{\omega'_B(x_j)}{2\pi}\right)^2.$$

Then it follows that

$$\sum_{j=1}^{n}\left(\left(\frac{\omega_A(x_j)}{2\pi}\right)\left(\frac{\omega_B(x_j)}{2\pi}\right)^2 + \left(\frac{\omega'_A(x_j)}{2\pi}\right)^2\left(\frac{\omega'_B(x_j)}{2\pi}\right)^2 + \left(\frac{\omega_A(x_j)}{2\pi}\right)^2\left(\frac{\omega'_B(x_j)}{2\pi}\right)^2 + \left(\frac{\omega'_A(x_j)}{2\pi}\right)^2\left(\frac{\omega_B(x_j)}{2\pi}\right)^2\right)^{\frac{1}{2}}$$

$$\geq \sum_{j=1}^{n}\left(\frac{\omega_A(x_j)}{2\pi}\frac{\omega_B(x_j)}{2\pi} + \frac{\omega'_A(x_j)}{2\pi}\frac{\omega'_B(x_j)}{2\pi}\right).$$

Hence,

$$\left(\sum_{j=1}^{n}\left\{\left(\frac{\omega_A(x_j)}{2\pi}\right)^2 + \left(\frac{\omega'_A(x_j)}{2\pi}\right)^2\right\}\sum_{i=1}^{n}\left\{\left(\frac{\omega_B(x_j)}{2\pi}\right)^2 + \left(\frac{\omega'_B(x_j)}{2\pi}\right)^2\right\}\right)^{\frac{1}{2}} \geq \sum_{j=1}^{n}\left(\frac{\omega_A(x_j)}{2\pi}\frac{\omega_B(x_j)}{2\pi} + \frac{\omega'_A(x_j)}{2\pi}\frac{\omega'_B(x_j)}{2\pi}\right)$$

$$\geq C_2(A,B).$$

Thus

$C_2(A,B) \leq \sqrt{E_2(A).E_2(B)}$.

2. For any two CFSs $A$ and $B$, we have,

$$K(A,B) = \frac{\frac{1}{2n}\left[\sum_{j=1}^{n}\left\{\left(r_A(x_j)r_B(x_j) + r'_A(x_j)r'_B(x_j)\right) + \frac{1}{4\pi^2}\left(\omega_A(x_j)\omega_B(x_j) + \omega'_A(x_j)\omega'_B(x_j)\right)\right\}\right]}{\sqrt{\frac{1}{2n}\left[\sum_{j=1}^{n}\left(r_A^2(x_j) + r_A'^2(x_j)\right) + \sum_{j=1}^{n}\left\{\left(\frac{\omega_A(x_j)}{2\pi}\right)^2 + \left(\frac{\omega'_A(x_j)}{2\pi}\right)^2\right\}\right]\cdot\frac{1}{2n}\left[\sum_{j=1}^{n}\left(r_B^2(x_j) + r_B'^2(x_j)\right) + \sum_{j=1}^{n}\left\{\left(\frac{\omega_B(x_j)}{2\pi}\right)^2 + \left(\frac{\omega'_B(x_j)}{2\pi}\right)^2\right\}\right]}}$$

$$= \frac{\frac{1}{2n}\left[\sum_{j=1}^{n}\left\{\left(r_B(x_j)r_A(x_j) + r'_B(x_j)r'_A(x_j)\right) + \frac{1}{4\pi^2}\left(\omega_B(x_j)\omega_A(x_j) + \omega'_{AB}(x_j)\omega'_A(x_j)\right)\right\}\right]}{\sqrt{\frac{1}{2n}\left[\sum_{j=1}^{n}\left(r_B^2(x_j) + r_B'^2(x_j)\right) + \sum_{j=1}^{n}\left\{\left(\frac{\omega_B(x_j)}{2\pi}\right)^2 + \left(\frac{\omega'_B(x_j)}{2\pi}\right)^2\right\}\right]\cdot\frac{1}{2n}\left[\sum_{j=1}^{n}\left(r_A^2(x_j) + r_A'^2(x_j)\right) + \sum_{j=1}^{n}\left\{\left(\frac{\omega_A(x_j)}{2\pi}\right)^2 + \left(\frac{\omega'_A(x_j)}{2\pi}\right)^2\right\}\right]}}$$

$= K(B,A)$.

3. If $A = B$, then $r_A(x_j) = r_B(x_j)$ and $\omega_A(x_j) = \omega_B(x_j)$ for all $j$, then from the equation (2) and (3) we have

$C(A,A) = \frac{1}{2n}\left[\sum_{j=1}^{n}\left\{\left(r_A^2(x_j) + r_A'^2(x_j)\right) + \frac{1}{4\pi^2}\left(\omega_A^2(x_j) + \omega_A'^2(x_j)\right)\right\}\right]$,

$E(A) = \frac{1}{2n}\left[\sum_{j=1}^{n}\left\{\left(r_A^2(x_j) + r_A'^2(x_j)\right) + \frac{1}{4\pi^2}\left(\omega_A^2(x_j) + \omega_A'^2(x_j)\right)\right\}\right]$.

Hence from (1), we have

$$K(A,A) = \frac{C(A,A)}{\sqrt{E(A).E(A)}} = 1.$$

**Definition 3.** Let $A$ and $B$ be two CFSs on $X$ and $\rho = \{\rho_1, \rho_2, \cdots \rho_n\}^T$ be the weighting vector of $x_j (j = 1, 2, \cdots n)$, where $x_j$ and $\sum_{j=1}^{n} \rho_j = 1$. Then the weighted correlation coefficient between $A$ and $B$ is denoted by $WK(A, B)$, is defined as

$$WK(A,B) = \frac{WC(A,B)}{\sqrt{WE(A).WE(B)}} \quad (5)$$

where,

$$WC(A,B) = \frac{1}{2n}\left[\sum_{j=1}^{n} \rho_j \left\{\left(r_A(x_j) r_B(x_j) + r'_A(x_j)\mu'_B(x_j)\right) + \frac{1}{4\pi^2}\left(\omega_A(x_j)\omega_B(x_j) + \omega'_A(x_j)\omega'_B(x_j)\right)\right\}\right] \quad (6)$$

$$WE(A) = \frac{1}{2n}\left[\sum_{j=1}^{n} \rho_j \left\{\left(r_A^2(x_j) + r_A'^2(x_j)\right) + \frac{1}{4\pi^2}\left(\omega_A^2(x_j) + \omega_A'^2(x_j)\right)\right\}\right] \quad (7)$$

$$WE(B) = \frac{1}{2n}\left[\sum_{j=1}^{n} \rho_j \left\{\left(r_B^2(x_j) + r_B'^2(x_j)\right) + \frac{1}{4\pi^2}\left(\omega_B^2(x_j) + \omega_B'^2(x_j)\right)\right\}\right] \quad (8)$$

**Example 2.** Consider the example 1. Let $\rho = \{0.4, 0.2, 0.25, 0.15\}^T$ be the weighting vector of $x_j$ $(j = 1, 2, 3, 4)$, then by the equations (6), (7) and (8) we have

$$WC(A,B) = \frac{1}{2n}\left[\sum_{j=1}^{n} \rho_j \left\{\left(r_A(x_j) r_B(x_j) + r'_A(x_j)\mu'_B(x_j)\right) + \frac{1}{4\pi^2}\left(\omega_A(x_j)\omega_B(x_j) + \omega'_A(x_j)\omega'_B(x_j)\right)\right\}\right]$$

$$= \frac{1}{2\times 4}\Big[0.4 \times \Big((0.8 \times 0.4 + 0.2 \times 0.6) + \frac{1}{4\pi^2}\big(2\pi(0.5) \times 2\pi(0.5) + 2\pi(0.5) \times 2\pi(0.5)\big)\Big) + 0.2 \times \Big((0.7 \times 0.8 + 0.3 \times 0.2) + \frac{1}{4\pi^2}\big(2\pi(0.4) \times 2\pi(0.6) + 2\pi(0.6) \times 2\pi(0.4)\big)\Big) + 0.25 \times \Big((0.6 \times 0.5 + 0.4 \times 0.5) + \frac{1}{4\pi^2}\big(2\pi(0.3) \times 2\pi(0.4) + 2\pi(0.7) \times 2\pi(0.6)\big)\Big) + 0.15 \times \Big((0.9 \times 0.3 + 0.1 \times 0.7) + \frac{1}{4\pi^2}\big(2\pi(0.8) \times 2\pi(0.2) + 2\pi(0.2) \times 2\pi(0.8)\big)\Big)\Big]$$

$$= 0.116$$

$$WE(A) = \frac{1}{2n}\left[\sum_{j=1}^{n} \rho_j \left\{\left(r_A^2(x_j) + r_A'^2(x_j)\right) + \frac{1}{4\pi^2}\left(\omega_A^2(x_j) + \omega_A'^2(x_j)\right)\right\}\right]$$

$$= \frac{1}{2\times 4}\Big[0.4 \times \Big((0.8)^2 + (0.2)^2 + \frac{1}{4\pi^2}((2\pi \times 0.5)^2 + (2\pi \times 0.5)^2)\Big) + 0.2 \times \Big((0.7)^2 + (0.3)^2 + \frac{1}{4\pi^2}((2\pi \times 0.4)^2 + (2\pi \times 0.6)^2)\Big) + 0.25 \times \Big((0.6)^2 + (0.4)^2 + \frac{1}{4\pi^2}((2\pi \times 0.3)^2 + (2\pi \times 0.7)^2)\Big) + 0.15 \times \Big((0.3)^2 + (0.7)^2 + \frac{1}{4\pi^2}((2\pi \times 0.8)^2 + (2\pi \times 0.2)^2)\Big)\Big]$$

$$=0.1445$$

$$WE(B) = \frac{1}{2n}\left[\sum_{j=1}^{n} \rho_j \left\{\left(r_B^2(x_j) + r_B'^2(x_j)\right) + \frac{1}{4\pi^2}\left(\omega_B^2(x_j) + \omega_B'^2(x_j)\right)\right\}\right]$$

$$= \frac{1}{2\times 4}\Big[0.4 \times \Big((0.4)^2 + (0.6)^2 + \frac{1}{4\pi^2}((2\pi \times 0.5)^2 + (2\pi \times 0.5)^2)\Big) + 0.2 \times \Big((0.8)^2 + (0.2)^2 + \frac{1}{4\pi^2}((2\pi \times 0.6)^2 + (2\pi \times 0.4)^2)\Big) + 0.25 \times \Big((0.5)^2 + (0.5)^2 + \frac{1}{4\pi^2}((2\pi \times 0.4)^2 + (2\pi \times 0.6)^2)\Big) + 0.15 \times \Big((0.9)^2 + (0.1)^2 + \frac{1}{4\pi^2}((2\pi \times 0.2)^2 + (2\pi \times 0.8)^2)\Big)\Big]$$

$$=0.141$$

Hence from the equation (5), we have

$$WK(A,B) = \frac{WC(A,B)}{\sqrt{WE(A).WE(B)}}$$

$$= \frac{0.116}{\sqrt{0.1445 \times 0141}}$$

$$= 0.83.$$

**Theorem 2.** Let $A$ and $B$ be two CFSs on $X$. If $\rho = \{\rho_1, \rho_2, \cdots \rho_n\}^T$ is the weighting vector of $x_j (j = 1, 2, \cdots n)$, with $x_j$ and $\sum_{j=1}^{n} \rho_j = 1$, then the weighted correlation coefficient $WK(A, B)$satisfies the following properties:

1. $0 \le WK(A, B) \le 1$,
2. $WK(A, B) = WK(B, A)$,
3. If $A = B$, then $WK(A, B) = 1$.

**Proof:** Similar as the proof of the theorem 1.

**Definition 4.**Let $A = \{(x, r_A(x). e^{i\omega_A(x)}) : x \in X\}$ and $B = \{(x, r_B(x). e^{i\omega_B(x)}) : x \in X\}$ be two CFSs on $X$. Then the max correlation coefficient between $A$ and $B$ is denoted by $K_1(A, B)$, is defined as

$$K_1(A, B) = \frac{C(A,B)}{max\{E(A),E(B)\}} \tag{9}$$

where,$C(A, B)$, $E(A)$ and $E(B)$ are as the definition 1.

**Example 3**.Considering the examples 1, then we have

$$\begin{aligned} K_1(A, B) &= \frac{C(A,B)}{max\{E(A),E(B)\}} \\ &= \frac{0.116}{max\{0.1445\times 0.141\}} \\ &= \frac{0.116}{0.1445} \\ &= 0.806. \end{aligned}$$

**Theorem 3.** Let $A$ and $B$ be two CFSs on $X$. Then the max correlation coefficient $K_1(A, B)$satisfies the following properties:

1. $0 \le K_1(A, B) \le 1$,
2. $K_1(A, B) = K_1(B, A)$,
3. If $A = B$, then $K_1(A, B) = 1$.

**Proof.** Let $A = \{(x, r_A(x). e^{i\omega_A(x)}) : x \in X\}$ and $B = \{(x, r_B(x). e^{i\omega_B(x)}) : x \in X\}$ be two CFSs on $X$. Then we have,

1. Since $C(A, B) \ge 0$, we need to only show that

$$C(A, B) \le \max\{E(A). E(B)\}.$$

In the proof of theorem 1, we have already proof that $C(A, B) \le \sqrt{E(A). E(B)}$, but by Schwarz inequality we have

$$C(A, B) \le \sqrt{E(A). E(B)} \le \max\{E(A). E(B)\}.$$

Thus

$$C(A, B) \le \max\{E(A). E(B)\}.$$

2. For any two CFSs $A$ and $B$, we have,

$$\begin{aligned} &K_1(A, B) \\ &= \frac{\frac{1}{2n}\left[\sum_{j=1}^{n}\left\{\left(r_A(x_j)r_B(x_j) + r'_A(x_j)r'_B(x_j)\right) + \frac{1}{4\pi^2}\left(\omega_A(x_j)\omega_B(x_j) + \omega'_A(x_j)\omega'_B(x_j)\right)\right\}\right]}{\max\left\{\frac{1}{2n}\left[\sum_{j=1}^{n}\left(r_A^2(x_j) + r_A'^2(x_j)\right) + \sum_{j=1}^{n}\left\{\left(\frac{\omega_A(x_j)}{2\pi}\right)^2 + \left(\frac{\omega'_A(x_j)}{2\pi}\right)^2\right\}\right], \frac{1}{2n}\left[\sum_{j=1}^{n}\left(r_B^2(x_j) + r_B'^2(x_j)\right) + \sum_{j=1}^{n}\left\{\left(\frac{\omega_B(x_j)}{2\pi}\right)^2 + \left(\frac{\omega'_B(x_j)}{2\pi}\right)^2\right\}\right]\right\}} \\ &= \frac{\frac{1}{2n}\left[\sum_{j=1}^{n}\left\{\left(r_B(x_j)r_A(x_j) + r'_B(x_j)r'_A(x_j)\right) + \frac{1}{4\pi^2}\left(\omega_B(x_j)\omega_A(x_j) + \omega'_{AB}(x_j)\omega'_A(x_j)\right)\right\}\right]}{\max\left\{\frac{1}{2n}\left[\sum_{j=1}^{n}\left(r_B^2(x_j) + r_B'^2(x_j)\right) + \sum_{j=1}^{n}\left\{\left(\frac{\omega_B(x_j)}{2\pi}\right)^2 + \left(\frac{\omega'_B(x_j)}{2\pi}\right)^2\right\}\right], \frac{1}{2n}\left[\sum_{j=1}^{n}\left(r_A^2(x_j) + r_A'^2(x_j)\right) + \sum_{j=1}^{n}\left\{\left(\frac{\omega_A(x_j)}{2\pi}\right)^2 + \left(\frac{\omega'_A(x_j)}{2\pi}\right)^2\right\}\right]\right\}} \\ &= K_1(B, A). \end{aligned}$$

3. If $A = B$, then $r_A(x_j) = r_B(x_j)$ and $\omega_A(x_j) = \omega_B(x_j)$ for all $j$, then from the equation (2) and (3) we have

$$C(A, A) = \frac{1}{2n}\left[\sum_{j=1}^{n}\left\{\left(r_A^2(x_j) + r_A'^2(x_j)\right) + \frac{1}{4\pi^2}\left(\omega_A^2(x_j) + \omega_A'^2(x_j)\right)\right\}\right],$$

$E(A) = \frac{1}{2n}\left[\sum_{j=1}^{n}\left\{\left(r_A^2(x_j) + r_A'^2(x_j)\right) + \frac{1}{4\pi^2}\left(\omega_A^2(x_j) + \omega_A'^2(x_j)\right)\right\}\right].$

Hence from (1), we have

$$K_1(A, A) = \frac{C(A,A)}{\max\{E(A).E(A)\}} = 1.$$

**Definition 5:** Let $A$ and $B$ be two CFSs on $X$ and $\rho = \{\rho_1, \rho_2, \cdots \rho_n\}^T$ be the weighting vector of $x_j (j = 1, 2, \cdots n)$, where $x_j$ and $\sum_{j=1}^{n} \rho_j = 1$. Then the weighted max correlation coefficient between $A$ and $B$ is denoted by $WK_1(A, B)$, is defined as

$$WK_1(A, B) = \frac{WC(A,B)}{max\{WE(A), W\ E(B)\}} \tag{10}$$

where,$WC(A, B)$, $WE(A)$ and $WE(B)$ are as the definition 3.

**Example 4.** Consider the example 2, and then we have

$$WK_1(A, B) = \frac{WC(A,B)}{max\{WE(A), WE(B)\}}$$
$$= \frac{0.4675}{max\{0.58, 0.5925\}}$$
$$= \frac{0.4675}{0.58}$$
$$= 0.806.$$

**Theorem 4.** Let $A$ and $B$ be two CFSs on $X$. Then the correlation coefficient $WK_1(A, B)$ satisfies the following properties:

1. $0 \leq WK_1(A, B) \leq 1$,
2. $WK_1(A, B) = WK_1(B, A)$,
3. If $A = B$, then $WK_1(A, B) = 1$.

**Proof:** Similar as the proof of the theorem 3.

## 4. Application

In this section, we propose the application of our proposed correlation coefficient in pattern recognition problem.

### 4.1 Pattern Recognition

The main object of pattern recognition is to decide how closely related some unknown patterns to an ideal pattern. We use correlation coefficient to find a closely related pattern from some unknown pattern to an ideal pattern.

Suppose there is an ideal pattern which is expressed as a complex fuzzy set

$$A = \left\{\left(x_j, r_A(x_j).\, e^{i\omega_A(x_j)}\right) : x_j \in X\right\} (j = 1, 2, \cdots n).$$

Let $B_r = \left\{\left(x_j, r_A(x_j).\, e^{i\omega_A(x_j)}\right) : x_j \in X\right\} (j = 1, 2, \cdots n), r = (1, 2, \cdots m)$ be some unknown patterns that to be recognized. Let $\rho = (\rho_1, \rho_2, \cdots, \rho_n)^T$ be the weighting vector of $x_j (j = 1, 2, \cdots, n)$ such that $\rho_j \in [0, 1]$, $(j = 1, 2, \cdots, n)$ and $\sum_{j=1}^{n} \rho_j = 1$.The objective is one of the patterns of $B_1, B_2, \cdots B_m$ is closely related to ideal pattern $A$ with the help of proposed correlation coefficient methods.

**Example 5.** We consider a simple pattern recognition problem concerning six digital images. The aim of the problem is to determine which one of the sample images belongs to ideal image. Let $X = \{x_1 = small, x_2 = medium, x_3 = large, x_4 = very\ large\}$ be the set of universe. Let a complex fuzzy set $A$ on $X$ as an ideal pattern and complex fuzzy sets $B_r$; $r = 1, 2, 3, 4, 5$on $X$ as unknown patterns. In these complex fuzzy sets, the amplitude and phase terms represent the degree of belongingness and the timestamp of the images. The complex of fuzzy sets of an ideal pattern $A$ and unknown patterns $B_r$; $r = 1, 2, 3, 4, 5$on $X$ is given in the table 1.

Table 1: The data sets of ideal pattern and unknown patterns

| | $B_1$ | $B_2$ | $B_3$ | $B_4$ | $B_5$ | $A$ |
|---|---|---|---|---|---|---|
| $x_1$ | $0.6e^{i\,2\pi(0.7)}$ | $0.4e^{i\,2\pi(0.2)}$ | $0.7e^{i\,2\pi(0.7)}$ | $0.7e^{i\,2\pi(0.6)}$ | $0.2e^{i\,2\pi(0.8)}$ | $0.7e^{i\,2\pi(0.5)}$ |
| $x_2$ | $0.9e^{i\,2\pi(0.8)}$ | $0.5e^{i\,2\pi(0.3)}$ | $0.4e^{i\,2\pi(0.6)}$ | $0.4e^{i\,2\pi(0.9)}$ | $0.7e^{i\,2\pi(0.3)}$ | $0.4e^{i\,2\pi(0.6)}$ |
| $x_3$ | $0.5e^{i\,2\pi(0.4)}$ | $0.6e^{i\,2\pi(0.4)}$ | $0.7e^{i\,2\pi(0.7)}$ | $0.7e^{i\,2\pi(0.7)}$ | $0.6e^{i\,2\pi(0.5)}$ | $0.5e^{i\,2\pi(0.5)}$ |
| $x_4$ | $0.6e^{i\,2\pi(0.4)}$ | $0.8e^{i\,2\pi(0.6)}$ | $0.6e^{i\,2\pi(0.5)}$ | $0.5e^{i\,2\pi(0.3)}$ | $0.6e^{i\,2\pi(0.5)}$ | $0.8e^{i\,2\pi(0.7)}$ |

The weight vector of $x_j (j = 1, 2, 3, 4)$ is: $\rho = (0.30, 0.35, 0.15, 0.20)$.

The correlation coefficient between $A$ and $B_r$; $r = 1, 2,3, 4, 5$ by applying proposed methods is given in the table 2.

Table 2: The Correlation coefficients between $A$ and $B_r$; $r = 1,2,3,4,5$

| Correlation coefficients | $(A, B_1)$ | $(A, B_2)$ | $(A, B_3)$ | $(A, B_4)$ | $(A, B_5)$ |
|---|---|---|---|---|---|
| $K(A, B_r)$ | 0.880 | 0.9428 | 0.9604 | 0.9269 | 0.8430 |
| $WK(A, B_r)$ | 0.8513 | 0.9316 | 0.9703 | 0.9399 | 0.8150 |
| $K_1(A, B_r)$ | 0.8603 | 0.9411 | 0.9500 | 0.9269 | 0.8340 |
| $WK_1(A, B_r)$ | 0.8101 | 0.9279 | 0.9598 | 0.9354 | 0.7944 |

Ranking of $B_r( r = 1, 2, 3, 4, 5)$ in accordance with the maximum values of the correlation coefficients are given by the table 3.

Table 3: Ranking of $B_r( r = 1, 2, 3, 4, 5)$

| Operators | Ranking | Best alternatives |
|---|---|---|
| $K(A, B_r)$ | $B_3 \succ B_2 \succ B_4 \succ B_1 \succ B_5$ | $B_3$ |
| $WK(A, B_r)$ | $B_3 \succ B_2 \succ B_4 \succ B_1 \succ B_5$ | $B_3$ |
| $K_1(A, B_r)$ | $B_3 \succ B_4 \succ B_2 \succ B_1 \succ B_5$ | $B_3$ |
| $WK_1(A, B_r)$ | $B_3 \succ B_2 \succ B_4 \succ B_1 \succ B_5$ | $B_3$ |

Therefore, it can be concluded that sample $B_3$ should belong to image $A$.

## 5. Comparison Studies

Since there is no other existing correlation coefficient method of complex fuzzy sets, so in this section, we discuss some comparative analysis with distance measure of complex fuzzy sets, as proposed by Zhang [15] and Alkouri [16]. The detailed analysis are given as below:

Let $A = \{(x, r_A(x). e^{i\omega_A(x)}): x \in X\}$ and $B = \{(x, r_B(x). e^{i\omega_B(x)}): x \in X\}$ be two CFSs on $X$. Then distance measure proposed by Zhang [15] and Alkouri [16] are given respectively as

$$D_Z(A,B) = max\left(\sup_{x\in X}|r_A(x) - r_B(x)|, \frac{1}{2\pi}\sup_{x\in X}|\omega_A(x) - \omega_B(x)|\right) \quad (11)$$

$$D_{HA}(A,B) = \frac{1}{2}\left(\sum_{j=1}^{n}|r_A(x_j) - r_B(x_j)| + \frac{1}{2\pi}\sum_{j=1}^{n}|\omega_A(x_j) - \omega_B(x_j)|\right) \quad (12)$$

$$D_{NHA}(A,B) = \frac{1}{2n}\left(\sum_{j=1}^{n}|r_A(x_j) - r_B(x_j)| + \frac{1}{2\pi}\sum_{j=1}^{n}|\omega_A(x_j) - \omega_B(x_j)|\right) \quad (13)$$

Table 4: The distance measures between $A$ and $B_r$; $i = 1,2,3,4,5$

| Distance Measures | $(A,B_1)$ | $(A,B_2)$ | $(A,B_3)$ | $(A,B_4)$ | $(A,B_5)$ |
|---|---|---|---|---|---|
| $D_Z(A,B)$ | 0.5 | 0.3 | 0.2 | 0.4 | 0.5 |
| $D_{HA}(A,B)$ | 0.8 | 0.65 | 0.5 | 0.9 | 0.95 |
| $D_{NHA}(A,B)$ | 0.2 | 0.1625 | 0.125 | 0.225 | 0.2375 |

Ranking of $B_r(\, r = 1,2,3,4,5)$ in accordance with the minimum values of the distance measures are given by the table 5.

Table-5: Ranking of $B_r(\, r = 1,2,3,4,5)$

| Operators | Ranking | Best alternatives |
|---|---|---|
| $D_Z(A,B)$ | $B_3 \succ B_2 \succ B_4 \succ B_1 = B_5$ | $B_3$ |
| $D_{HA}(A,B)$ | $B_3 \succ B_2 \succ B_1 \succ B_4 \succ B_5$ | $B_3$ |
| $D_{NHA}(A,B)$ | $B_3 \succ B_2 \succ B_1 \succ B_{14} \succ B_5$ | $B_3$ |

Therefore, we can conclude that sample $B_3$again should belong to image $A$.

From these comparative studies, it is concluded that the best alternative obtained from our proposed method coincides with the existing methods.

## 6. Conclusions

This article intense on developing some properties and the notions of correlation coefficient for the Complex fuzzy set (CFS). This study extended the work of Chiang and Lin [10] concerning the correlation coefficient of FSs in a new context of CFSs, which were exposed to be better of modelling real-life applications than the FSs. Theoretical exploration of correlation coefficient for CFSs were pointed out. These give the better to understand about the behaviour of correlation coefficient of CFSs which are helpful to select proper settings for applications. The last section in this article, the concept of correlation coefficient of CFSs applied in pattern recognition.

## References

[1] L. A. Zadeh, Fuzzy sets, *Inform. Control,* 8: 338-353 (1965).

[2] D .Ramot et al., Complex fuzzy sets, *IEEE Trans Fuzzy Syst.,* 10(2): 171–186 (2002).

[3] T. Gerstenkorn and J. Manko, Correlation of intuitionistic fuzzy sets, *Fuzzy Sets and Systems,* 44(1): 39–43 (1991).

[4] D. H. Hong, S. K. Hwang, Correlation of intuitionistic fuzzy sets in probability spaces, *Fuzzy Sets Syst.,* 75: 77–81 (1995).

[5] W. L. Hung, J. W. Wu, Correlation of intuitionistic fuzzy sets by centroid method, *Inf. Sci.,* 144: 219–225 (2002).

[6] H. Bustince, P. Burillo, Correlation of interval-valued intuitionistic fuzzy sets, *Fuzzy Sets Syst.* 74: 237–244 (1995).

[7] W. Zeng, H. Li, Correlation coefficient of intuitionistic fuzzy sets, *J Ind Eng Int.* 3(5): 33–40 (2007).

[8] H. Garg Novel correlation coefficients under the intuitionistic multiplicative environment and their applications to decision - making process, *Journal of Industrial and Management Optimization;* 13(5):1–19 (2018).

[9] B. Liu, et al., A new correlation measure of the intuitionistic fuzzy sets, *J Intell. Fuzzy Syst*. 30: 1019– 1028 (2016).

[10] D. A. Chiang and N. P. Lin, Correlation of fuzzy sets*, Fuzzy Sets and Systems* 102(2): 221–226 (1999).

[11] Y Je, Correlation coefficient of dual hesitant fuzzy sets and its application to multiple attribute decision making, Appl. Math. Model. 38 (2): 659-666 (2014).

[12] C. C. Yang Correlation coefficient evaluation for the fuzzy interval data, *J. Bus. Res.* 69 (6): 2138-2144 (2016).

[13] O. I. Ban et al., Evaluation of hotel quality attributes importance through fuzzy correlation coefficient. *Technol, Econ. Dev. Econ*.l (4): 71-492 (2016).

[14] B. B. Chaudhuri and A. Bhattacharya, On correlation between two fuzzy sets, *Fuzzy Sets and Systems* 118: 447-456 (2001).

[15] Zhang et al., Operation properties and δ-equalities of complex fuzzy sets, *International Journal of Approximate Reasoning* 50: 1227–1249 (2009).

[16] A.U.M. Alkouri and A. R. Salleh, Linguistic variable, hedges and several distances on complex fuzzy sets, *Journal of Intelligent & Fuzzy Systems* 26: 2527–2535 (2014).