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Lagrangian Relaxation Method in Approximating the Pareto Front of Multiobjective Optimization Problems

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ABSTRACT

In this paper, we propose that the Lagrangian relaxation approach can be used to approximate the Pareto front of the multiobjective optimization problems. We introduce Lagrangian relaxation approach to solve scalarized subproblems. The scalarization is a technique employed to transform multiple objectives optimization problems into single-objective optimization problems so that existing optimization techniques are used to solve the problems. The relaxation approach exploits transformation and creates a Lagrangian problem in which some of the constraints are replaced from the original problem to make the problem easier to solve. The method is very effective when the problem is large scale and difficult to solve; this means if the problem has nonconvex and nonsmooth structure, then our proposed method efficiently solves the problem. We succeed in establishing proper Karush Kuhn-Tucker type necessary conditions for our proposed approach. We establish the relation between our proposed approach and the well-known existing approach weighted-sum scalarization methods. We conduct extensive numerical experiments and demonstrated the advantages of the proposed method of adopting a test problem.

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1. Introduction

Lagrangian relaxation method is one of the useful approaches that is being used to solve large scale optimization problems. This can be regarded as an extension of the nonlinear programming Lagrange multiplier method. In the 1970s, Lagrangian relaxation grew up as a theoretical concept to a tool that is the backbone of a number of large-scale applications. There were several surveys of Lagrangian relaxation method conducted by Fisher [1], Geoffrion [2], and an excellent textbook treatment by Shapiro [3]. How to put a Lagrangian relaxation approach in practice has been analyzed in Fisher [1] and Held et al. [4].

In the real world, many problems have been modeled as optimization problems whose objective function includes a certain number of reasonable performance measures. We have to choose at least one individual better off, without making any other individual worse off. These types of problems we call a multiobjective optimization problem. One of the popular methods to solve multiobjective optimization problem is scalarization method (see, [5-8]). The scalarization approach is applied to transform multiple objectives into a single objective so that existing optimization techniques are used to solve

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the problems. In this paper, we introduce this Lagrangian relaxation approach to solve scalarized subproblems to approximate the Pareto front. The relaxation approach exploits transformation and creates a Lagrangian problem in which the complicating constraints are replaced. In our analysis, in the scalarization process, the objective functions which play the role as constraints are usually considered as complex constraints. Therefore, the relaxation approach is used to replace these objective-constraints from scalarized subproblems. The method is very effective when the problem is complex and difficult to solve; this means if the problem is either nonconvex or nonsmooth or both, then our proposed method efficiently solves the problem and generate the Pareto front. The proposed method could be more useful in the area of the control optimization problem, as in this area problems often modeled as a nonsmooth problem over time. And the existing solver cannot handle the problem properly because of the structure of the problem. In this situation, our proposed method, which is the augmented form of the original problem, works well to fit solvers to approximate solutions. We also show that proper Karush Kuhn-Tucker type necessary conditions hold for our proposed approach. Moreover, we established the relation between our proposed approach and the well-known existing approach weighted-sum scalarization methods. In the end, we conduct extensive numerical experiments and demonstrated the efficiency of the proposed method using a test problem.

2. Multiobjective optimization problems

In this section, we introduce some notations and definitions of multiobjective optimization problems, which are used throughout the paper. Let E_n be n-dimensional Euclidean space.
For $\boldsymbol{x}, \boldsymbol{y} \in E_n$, we use the following conventions.

$$\boldsymbol{x} \geqq \boldsymbol{y}, \text{ iff } x_i \geqq y_i, \text{ i=1,…,n,}$$
$$\boldsymbol{x} \geq \boldsymbol{y}, \text{ iff } \boldsymbol{x} \geqq \boldsymbol{y} \text{ and } \boldsymbol{x} \neq \boldsymbol{y},$$
$$\boldsymbol{x} > \boldsymbol{y}, \text{ iff } x_i > y_i \text{ i=1,…,n.}$$

Now, we consider the following multiobjective optimization Problem P:

$$min\, Z = f\,(\boldsymbol{x}), \text{ subject to the set X:}$$
$$\bar{\boldsymbol{x}} \in X = \{\boldsymbol{x} \in E_n | \boldsymbol{g}(\boldsymbol{x}) \leqq 0\}.$$

Assume that, $f: E_n \to E_l$ and $g: E_n \to E_m$ be continuously differentiable vector-valued functions.

2.1 Pareto optimality /efficiency/non-superiority

We consider a solution $\bar{\boldsymbol{x}} \in X$ of Problem P. Now, $\bar{\boldsymbol{x}}$ is an optimal solution if and only if every objective function $f(\boldsymbol{x}) \equiv (f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), \ldots, f_l(\boldsymbol{x}))$ attains its minimum value. This appears that the conflicting nature of the objectives, an optimal solution that simultaneously minimizes all the objectives is usually not attainable, that is, the definition of optimality is too stronger to be used. The extension of the definition of an optimal solution is thus necessary and desirable. Many broader definitions of optimal solutions are used in the literature: these are Pareto optimal point [7, 8], vector minimum point [7], efficient point [9,10] and non-superiority [9]. In our analysis, we refer to this point as a Pareto point or an efficient point. The front that contains Pareto points is called the Pareto front.

Definition 2.1 A point $\bar{\mathbf{x}} \quad \mathrm{X}$ is called an *efficient* solution to Problem P if there is no $\mathbf{x} \quad \mathrm{X}$ such that $\boldsymbol{f}(\boldsymbol{x}) \leq \boldsymbol{f}(\bar{\boldsymbol{x}})$.

A point $\bar{\mathbf{x}} \quad \mathrm{X}$ is called a *weak efficient* solution to Problem P if there is no $\mathbf{x} \quad \mathrm{X}$ such that $f(\boldsymbol{x}) < f(\bar{\boldsymbol{x}})$.

Because of ordering relations, we can get a problem where all points are efficient solutions or the problem does not have any efficient point at all.

2.2 Scalarization method

A multiobjective problem is often solved by combining its multiple objectives into one single-objective scalar function. This approach is, in general known as the scalarization method. To obtain a set of efficient points, a set of weighting vectors is used which would result in a set of single-objective subproblems. This is the reason why such methods are called decomposition-based. Because the employed strategy is to decompose a complex problem into a set of simpler ones. Simpler in this context does not necessarily mean easier to solve, it means that it is straightforward to apply standard methods to solve subproblems. One of the well-known scalarization techniques is k^{th} objective weighted constraint problem introduced in [6,11]. The method is applicable not only to the problems with disconnected Pareto front but also to the problems with a disconnected feasible set under the mild assumptions that the objective functions are continuous and bounded from below with a known lower bound. For each fixed k, the k$^{\text{th}}$ objective is minimized while the other weighted objective functions are incorporated as constraints. The Problem (BMOP) introduced in [6,11] structure is as follows:

$$\min w_k f_k(x),$$

subject to

$$w_i f_i(x) \leq w_k f_k(x), i = 1, \ldots, l, i \neq k,$$
$$w_i > 0,\ \textstyle\sum_{i=1}^{l} w_i = 1,$$
$$g_j(x) \leq 0,\ j = 1, \ldots, n.$$

If $\bar{x}$ is a weak efficient point to the original Problem P, then $\bar{x}$ is the solution of the (BMOP) for each fixed k. On the other hand, a point $\bar{x}$ solves all subproblems of (BMOP), then it is an efficient solution of the original Problem P. This is a strong condition and it can be relaxed by setting x_k, $k = 1, \ldots, l$ solves the k-subproblems of (BMOP) and if Proposition 3.3 in [6] holds, then x_k, $k = 1, \ldots, l$ are weak efficient solutions of the original Problem P.

We recall here another popular scalarization method to solve Problem P which is known as the weighted sum approach introduced in [12,13].

The method (GMOP) is as follows

$$\min \textstyle\sum_{i=1}^{l} w_i f_i(x),$$

subject to

$$g_j(x) \leq 0,\ j = 1, \ldots, n,$$
$$w_i > 0,\ \textstyle\sum_{i=1}^{l} w_i = 1.$$

Every solution of Problem (GMOP) is a weakly efficient point [8], and this fact is used to construct an approximation of the Pareto front. Note that the main advantage of this method is that it is very efficient and easy to implement. However, a drawback of this approach is it cannot approximate the Pareto point which lies in the nonconvex part of the Pareto front.

3. Lagrangian relaxation in multiobjective optimization problem

Many hard integer programming problems can be regarded as easy problems if it is possible to relax some of the side constraints from a constraint set. Fisher [1] proposes a method that dualizes complex constraints produces a Lagrangian problem that is straight forward to solve and whose optimal value is a lower bound (for minimization problems) on the optimal value of the original problem. This approach has led to dramatically improved algorithms for many important problems in the areas of routing, location, scheduling, assignment and set covering. We extend this Fisher's [1] Lagrangian relaxation approach to solve multiobjective optimization problems.

We provide a Lagrangian relaxation technique to solve the subproblems of (BMOP), start with relaxing the constraints

$$w_i f_i(x) \leq w_k f_k(x), i = 1, \ldots, l,\ i \neq k,$$

We define an $(l-1)$ vector of nonnegative multipliers $p \in E_l$, $p \geq 0$ and added the nonnegative term

$$p^i(w_i f_i(x) - w_k f_k(x)), i = 1, \ldots, l,\ i \neq k,$$

to the objective function of (BMOP). Therefore, the mathematical formulation of the revised k$^{\text{th}}$ subproblem is (LR_{BMOP})

$$\min w_k f_k(x) + \textstyle\sum_{i=1}^{l-1} p^i [w_i f_i(x) - w_k f_k(x)],\ i \neq k,$$

subject to

$$w_i > 0,\ \textstyle\sum_{i=1}^{l} w_i = 1,$$
$$g_j(x) \leq 0,\ j = 1, \ldots, n,$$
$$p \geq 0.$$

This can be reformulated as

$$\min\left(1 - \textstyle\sum_{i=1}^{l-1} p^i\right) w_k f_k(x) + \textstyle\sum_{i=1}^{l-1} p^i\ w_i f_i(x),\ i \neq k,$$

subject to

$$w_i > 0,\ \textstyle\sum_{i=1}^{l} w_i = 1,$$
$$g_j(x) \leq 0,\ j = 1, \ldots, n,$$
$$p \geq 0.$$

There are three major questions arise in designing the Lagrangian relaxation method.

a. Find the constraints set that should be relaxed from the problem?
b. Find the multipliers p^i that minimize the problems?

c. How to presume a good feasible solution to Problem P?

These fully depend on the specific problem.

3.1 Determining Lagrangian parameters p^i

Fisher suggested that the best choice of finding p^i is to solve the dual problem of (LR_{BMOP}). However, dualize the problem and then solve it to find the Lagrangian parameters p^i has a clear disadvantage as the procedure is computationally extensive. In our analysis, we use Held et al. [4] approach to find the parameters p^i which is as follows.
The subgradient method is used for setting dual variables. A detailed of this setting can be seen in [2]. We are here recalling the main steps to calculate p^i. It is required to determine a sequence of values for p^i that are

$$p^{k+1} = \max\{\, 0, p^k + t_k(Ax^k - b)\},$$

where t^k is the step-size and x^k are the optimal solution of (LR_{BMOP}).
A formula for t^k that has been used in practice is

$$t^k = \frac{\lambda_k(Z^* - Z_D(u_k))}{\left\|Ax^k - b\right\|^2},$$

Z^* is the objective value of the best-known feasible solution of (P). The sequence λ_k is determined from the interval $[0,2]$ and suitable λ_k is required if $Z(p^i)$ fails to increase.

4. Existence of Karush Kuhn-Tucker Conditions

In this section, we show that the solution of (LR_{BMOP}) for all k, and the regularity conditions GGRC guarantees proper Karush Kuhn-Tucker conditions (PKKT). Proper KKT conditions are said to hold if all the Lagrangian multipliers of the objective functions are positive (see, more details in [14,15]). To obtain (PKKT), a set of assumptions is required which contains the objective functions and the constraints. These assumptions are called regularity conditions. The following assumptions will be needed in the development.

Assumption A: Let $\bar{x}$ be an efficient point and the following regularity condition GGRC holds at $\bar{x}$.
(Generalized Guignard Regularity Conditions (GGRC))[11] Let $\bar{x}$ be a feasible point of (MOP). Then (MOP) is said to satisfy the GGRC if

$$L(Q;\bar{x}) \subseteq \bigcap_{i=1}^{l} clconvT(Q^i;\bar{x}),$$

where, $Q^i \equiv \{\boldsymbol{x} \in E_n | \boldsymbol{x} \in X,\ f_k(\boldsymbol{x}) \leq f_k(\bar{\boldsymbol{x}}),\ k = 1,2,\ldots,l \ \ and \ \ k \neq i\}$, and

$$Q \equiv \{\boldsymbol{x} \in E_n | \boldsymbol{x} \in X,\ \mathbf{f}(\boldsymbol{x}) \leq \boldsymbol{f}(\bar{\boldsymbol{x}})\}.$$

Theorem 4.1: Under Assumption A, let $\bar{x}$ be the solution of (LR_{BMOP}) for all k. Moreover, for every fixed k, all the multipliers associated with Problems (LR_{BMOP}) are positive.

Proof: Theorem 4.3.1 in [6] states that every weak efficient solution of Problem (MOP) is the solution of (BMOP) for all k. We also know that every efficient point is a weak efficient point. Therefore, efficient points are the solution of (BMOP) for all k. By the assumptions we have (see Theorem 4.4 in [6])

$$\sum_{i=1}^{l} u_i \nabla f_i(\bar{\boldsymbol{x}}) + \sum_{j=1}^{m} v_j \nabla g_j(\bar{\boldsymbol{x}}) = 0 \tag{4.1}$$
$$v_j g_j(\bar{\boldsymbol{x}}) = 0, j = 1, \ldots, m,$$
$$\boldsymbol{u} > 0, \boldsymbol{v} \geq 0.$$

We can rearrange (4.1) as

$$\frac{u_k}{w_k} w_k \nabla f_k(\bar{\boldsymbol{x}}) + \sum_{i=1, i\neq k}^{l} \frac{u_i}{w_i} w_i \nabla f_i(\bar{\boldsymbol{x}}) + \sum_{j=1}^{m} v_j \nabla g_j(\bar{\boldsymbol{x}}) = 0,$$

$$\left(\frac{u_k}{w_k} + \sum_{i=1, i\neq k}^{l} \frac{u_i}{w_i}\right) w_k \nabla f_k(\bar{\boldsymbol{x}}) + \sum_{i=1, i\neq k}^{l} \frac{u_i}{w_i} \left(w_i \nabla f_i(\bar{\boldsymbol{x}}) - w_k \nabla f_k(\bar{\boldsymbol{x}})\right) + \sum_{j=1}^{m} v_j \nabla g_j(\bar{\boldsymbol{x}}) = 0,$$

this implies,

$$\sum_{i=1}^{l} \frac{u_i}{w_i} w_k \nabla f_k(\bar{\boldsymbol{x}}) + \sum_{i=1, i\neq k}^{l} \frac{u_i}{w_i} \left(w_i \nabla f_i(\bar{\boldsymbol{x}}) - w_k \nabla f_k(\bar{\boldsymbol{x}})\right) + \sum_{j=1}^{m} v_j \nabla g_j(\bar{\boldsymbol{x}}) = 0,$$

therefore,

$$\sum_{i=1}^{l}\frac{u_i}{w_i}\ w_k\nabla f_k(\bar{\boldsymbol{x}})+\sum_{i=1,i\neq k}^{l}\ \sum_{i=1}^{l}\frac{u_i}{w_i}\frac{\frac{u_i}{w_i}}{\sum_{i=1}^{l}\frac{u_i}{w_i}}\left(w_i\nabla f_i(\bar{\boldsymbol{x}})-w_k\,\nabla f_k(\bar{\boldsymbol{x}})\right)+\sum_{j=1}^{m}v_j\nabla g_j(\bar{\boldsymbol{x}})=0. \tag{4.2}$$

Assume that $\lambda_i=\frac{u_i}{w_i}$ and $p^i=\frac{\frac{u_i}{w_i}}{\sum_{i=1}^{l}\frac{u_i}{w_i}}$. Since, $u_i>0$ and $w_i>0$ for $i=1,\dots,l$, thus $\lambda_i>0$ and hence, $p^i>0$. Let $\lambda=\sum_{i=1}^{l}\frac{u_i}{w_i}$, therefore the equation (4.2) becomes

$$\lambda\left(w_k\nabla f_k(\bar{\boldsymbol{x}})+\sum_{i=1,i\neq k}^{l}p^i\left(w_i\nabla f_i(\bar{\boldsymbol{x}})-w_k\,\nabla f_k(\bar{\boldsymbol{x}})\right)\right)+\sum_{j=1}^{m}v_j\nabla g_j(\bar{\boldsymbol{x}})=0.$$

We also have

$$v_jg_j(\bar{\boldsymbol{x}})=0,\qquad j=1,\dots,m.$$

Hence, $\lambda_i>0,\ i=1,\dots,l,$ as required.

Now, we establish the relation of the weighted sum scalarization (GMOP) and the Lagrangian relaxation form of k[th] objective weighted constraint problem (LR_{BMOP}).

Lemma 4.1 If there exists $w\in W$ such that $\bar{x}$ solves (WMOP), then there exists $p_i\geq 0$ for all $j\neq k$ such $\bar{x}$ solves (LR_{BMOP}).

Proof: Since, $\bar{x}$ solves (WMOP) for some $w\in W:=\{x\in E_l|w_i\geq 0,\sum_{i=1}^{l}w_i=1\}$, we have that $\sum_{i=1}^{l}w_i\nabla f_i(\bar{\boldsymbol{x}})\leq\sum_{i=1}^{l}w_i\nabla f_i(x)$, for all $x\in X$. (4.3)

We can rearrange (4.3) as

$$\frac{w_k}{u_k}u_k\nabla f_k(\bar{\boldsymbol{x}})+\sum_{i=1,i\neq k}^{l}\frac{w_i}{u_i}u_i\nabla f_i(\bar{\boldsymbol{x}})\leq\frac{w_k}{u_k}u_k\nabla f_k(x)+\sum_{i=1,i\neq k}^{l}\frac{w_i}{u_i}u_i\nabla f_i(x),$$

$$\left(\frac{w_k}{u_k}+\sum_{i=1,i\neq k}^{l}\frac{w_i}{u_i}\right)\ u_k\nabla f_k(\bar{\boldsymbol{x}})+\sum_{i=1,i\neq k}^{l}\frac{w_i}{u_i}\ \left(u_i\nabla f_i(\bar{\boldsymbol{x}})-u_i\nabla f_k(\bar{\boldsymbol{x}})\right)$$
$$\leq\left(\frac{w_k}{u_k}+\sum_{i=1,i\neq k}^{l}\frac{w_i}{u_i}\right)\ u_k\nabla f_k(x)+\sum_{i=1,i\neq k}^{l}\frac{w_i}{u_i}\ \left(u_i\nabla f_i(x)-u_i\nabla f_k(x)\right),$$

this implies,

$$\sum_{i=1}^{l}\frac{w_i}{u_i}\ u_k\nabla f_k(\bar{\boldsymbol{x}})+\sum_{i=1,i\neq k}^{l}\frac{w_i}{u_i}\ \left(u_i\nabla f_i(\bar{\boldsymbol{x}})-u_k\,\nabla f_k(\bar{\boldsymbol{x}})\right)\leq\sum_{i=1}^{l}\frac{w_i}{u_i}\ u_k\nabla f_k(x)+\sum_{i=1,i\neq k}^{l}\frac{w_i}{u_i}\ \left(u_i\nabla f_i(x)-u_k\,\nabla f_k(x)\right),$$

therefore,

$$\sum_{i=1}^{l}\frac{w_i}{u_i}\ u_k\nabla f_k(\bar{\boldsymbol{x}})+\sum_{i=1,i\neq k}^{l}\ \sum_{j=1}^{l}\frac{w_j}{u_j}\frac{\frac{w_i}{u_i}}{\sum_{j=1}^{l}\frac{w_j}{u_j}}\left(u_i\nabla f_i(\bar{\boldsymbol{x}})-u_k\,\nabla f_k(\bar{\boldsymbol{x}})\right)$$
$$\leq\sum_{i=1}^{l}\frac{w_i}{u_i}\ u_k\nabla f_k(x)+\sum_{i=1,i\neq k}^{l}\ \sum_{j=1}^{l}\frac{w_j}{u_j}\frac{\frac{w_i}{u_i}}{\sum_{j=1}^{l}\frac{w_j}{u_j}}\left(u_i\nabla f_i(x)-u_k\,\nabla f_k(x)\right). \tag{4.4}$$

Assume that $\lambda_i=\frac{w_i}{u_i}$ and $p^i=\frac{\frac{w_i}{u_i}}{\sum_{j=1}^{l}\frac{w_j}{u_j}}, for\ i=1,\dots,l.$ Since, $u_i>0$ and $w_i>0$ for $i=1,\dots,l,$ thus $\lambda_i>0$ and hence, $p^i>0$. Let $\lambda=\sum_{i=1}^{l}\frac{w_i}{u_i}$, therefore the equation (4.4) becomes

$$\lambda\left(u_k\nabla f_k(\bar{\boldsymbol{x}})+\sum_{i=1,i\neq k}^{l}p^i\left(u_i\nabla f_i(\bar{\boldsymbol{x}})-u\nabla f_k(\bar{\boldsymbol{x}})\right)\right)\leq\lambda\left(u_k\nabla f_k(x)+\sum_{i=1,i\neq k}^{l}p^i\left(u_i\nabla f_i(x)-u\nabla f_k(x)\right)\right).$$

Thus,

$$u_k\nabla f_k(\bar{\boldsymbol{x}})+\sum_{i=1,i\neq k}^{l}p_i\left(u_i\nabla f_i(\bar{\boldsymbol{x}})-u\nabla f_k(\bar{\boldsymbol{x}})\right)\leq u_k\nabla f_k(x)+\sum_{i=1,i\neq k}^{l}p^i\left(u_i\nabla f_i(x)-u\nabla f_k(x)\right).$$

Hence, $\bar{x}$ solves (LR_{BMOP}) as required.

5. Numerical Illustrations

In this section, we demonstrate employing a test problem that the proposed method (LR_{BMOP}) efficiently generate Pareto front of the Problem P. We also provide an algorithm to solve scalarization subproblems using Lagrangian relaxation techniques.

Test Problem 5.1

We take $n = l = 2$; Consider the problem

$$\min \{f_1(x), f_2(x)\}, \text{ here, } f_1(x) = x_1, \text{ and } f_2(x) = x_2,$$

$$\text{subject to } X = \{(x_1, x_2) | (x_1 - 1)^2 + (x_2 - 1)^2 - 1 \leq 0\}.$$

We now use the k^th objective scalarization approach (BMOP) and construct the auxiliary subproblems and respective Lagrangian relaxation approach to solve Test Problem 5.1. (see, Table 5.1)

Table 5.1: Scalarized subproblems.

k^{th} objective weighted constraint problem	Subproblems of Example 5.1	Lagrangian relaxation form of the subproblems
(Subproblem-1)	$\min w_1 f_1(x)$, subject to $w_2 f_2(x) \leq w_1 f_1(x)$, $w_i > 0, \sum_{i=1}^{2} w_i = 1$. $x \in X$.	$\min w_1 x_1 + p^2(w_2 x_2 - w_1 x_1)$, subject to $x \in X$.
(Subproblem-2)	$\min w_2 f_2(x)$, subject to $w_1 f_1(x) \leq w_2 f_2(x)$, $w_i > 0, \sum_{i=1}^{2} w_i = 1$, $x \in X$.	$\min w_2 x_2 + p^1(w_1 x_1 - w_2 x_2)$, subject to $x \in X$.

Algorithm (LR_{BMOP} for l = 2)

Step 1 (Input)

Choose $u = (u_1, u_2)$ which is a reference point used to pass a ray to identify the Pareto point on the front. Set the number of partitioned points (N+1).

Step 2 (Obtain outer endpoints of the Pareto front)

a. Solve the Lagrangian relaxation form of (Subproblems-1).
b. Solve the Lagrangian relaxation form of (Subproblems-2).

Evaluate $w = (w_0, w_f)$ as introduced in (Step 2 of Algorithm 3 in [11])

Step 3 (Generate a weight partition)

This step is the same as Step 3 of Algorithm 3 in [11].

Step 4 (Obtain Lagrangian parameter p^i)

Set, while $t^k > .0001$

Follow Step 4 (a and b) of Algorithm 3 in [11] and find optimum solution $\bar{x} = (x_1, x_2)$.

Evaluate t^k in (1) until it met the while condition. Find p^i that minimize (Subproblems-1 & 2).

Follow Step 4 (c) of Algorithm 3 in [11].

Step 5 Repeat the Steps 2-4 for all weight vectors w.

Step 6 Record Data.

Remark: 5.1

The implementation of Held et al. [4] algorithm is also quite computationally expensive, but we choose this approach in our analysis as it is easy to implement. The algorithms need to fix $0 \leq \lambda \leq 2$ consistently for each weight vector because of the complex procedure of determining Lagrangian multipliers p^1 and p^2. Otherwise, the algorithm does not find

Lagrangian multipliers (p^1 or p^2) for which the model converges to a solution. This is the drawback of Lagrangian relaxation techniques. However, the method has a clear advantage when the objective functions are nonconvex and nonsmooth. Figure 5.1 demonstrates the Pareto front obtained by the Lagrangian relaxation approach (Subproblems-1 & 2). The implementation of the algorithm required MATLAB programming software. It is also noted that we utilize a range of nonlinear solvers such as fmincon (sqp and interior point algorithms), 'Ipopt'[16] and 'SCIP'[17], and weight-generation algorithms [18] to solve each of the subproblems.

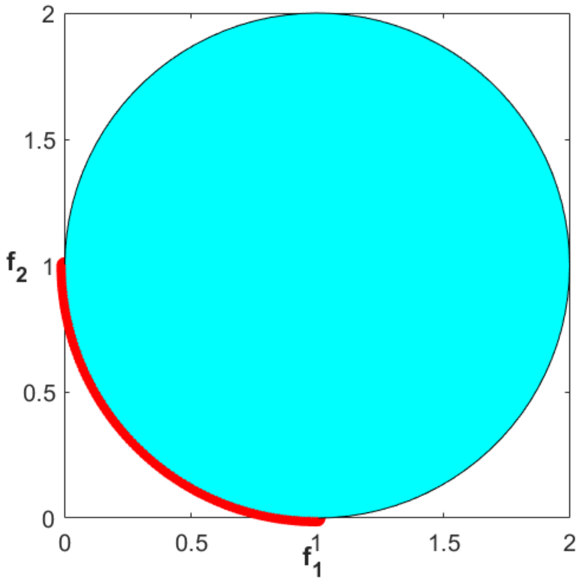

Figure 5.1: The circle (Red) depicts the efficient solution of Test Problem 5.1 which also approximates the Preto front.

6. Conclusions

We proposed a new approach to solve scalarization problems that formed from multiobjective optimization problems. Lagrangian relaxation technique has been used in the proposed approach. We have shown that the new approach effectively solves the problem and approximate the Pareto front efficiently. Moreover, the proposed method can be used in large scale problems such as when the problem has more than three objective functions, and when the functions are nonconvex and nonsmooth. We also have shown that the proposed approach guarantees the proper Karush Kuhn-Tucker conditions. Moreover, we prove that the proposed method is more general than the well-known weighted scalarization technique. The algorithm with numerical experiments conducted to test the efficiency of the proposed method.

References

[1]. Fisher, M. L., The Lagrangian Relaxation Method for Solving Integer Programming Problems, Management Science, 27(1) (1981), 1–18.
[2]. Geoffrion, A. M., Lagrangian Relaxation for Integer Programming, Mathematical Programming, 2 (1974), 82–114.
[3]. Shapiro, J. F., Mathematical Programming: Structures and Algorithms, John Wiley, New York, 1979.
[4]. Held, M., Wolfe, P., and Crowder, H. D., Validation of Subgradient Optimization, Math. Programming, 6 (1974), 62-88.
[5]. Burachik R. S., Kaya C. Y. and Rizvi, M. M., A New Scalarization Techniques and New Algorithms to Generate Pareto Fronts, SIAM Journal on Optimization (SIOPT), 27(2) (2017), 1010-1034.
[6]. Burachik R. S., C. Y. Kaya and Rizvi, M. M., A New Scalarization Technique to Approximate Pareto Fronts of Problems with Disconnected Feasible Sets, Journal of Optimization Theory and Applications, 162(2) (2014), 428-446.
[7]. Miettinen, K. M., Nonlinear multiobjective optimization, Boston: Kluwer Academic, 1999.
[8]. Chankong, V., and Haimes, Y. Y., Multiobjective decision making: Theory and Methodology, Amsterdam: North-Holland, 1983.
[9]. Wang, S. Y., A Note on Optimality Conditions in Multiobjective Programming, Systems Sci. and Math. Sci., 1 (1988), 184–190.
[10]. Bigi, G., Optimality and Lagrangian Regularity in Vector Optimization, Ph. D. Thesis, 2003.
[11]. Rizvi, M. M., New Optimality Conditions for Nonlinear Multiobjective Optimization Problems and New Scalarization Techniques for Constructing Pathological Pareto Fronts, Ph. D. Thesis, 2013.

[12]. Gass, S. and Saaty, T., The Computational Algorithm for the Parametric Objective Function, Naval Res. Logis. quart., 2 (1955), 39–45.

[13]. Zadeh, L., Optimality and Non-scalar-valued Performance Criteria, IEEE Transactions on Automatic Control. 8 (1963), 59–60.

[14]. Burachik R. S. and Rizvi, M. M., Proper Efficiency and Proper Karush-Kuhn-Tucker Conditions for Smooth Multiobjective Optimization Problems. Vietnam Journal of Mathematics, 42(4) (2014), 521-531.

[15]. Burachik R. S. and Rizvi, M. M., On Weak and Strong Kuhn-Tucker Conditions for Smooth Multiobjective Optimization. Journal of Optimization Theory and Applications, 155(2) (2012), 477-491.

[16]. Wachter, A. and Biegler, L. T., On the Implementation of a Primal-dual Interior Point Filter Line Search Algorithm for Large-scale Nonlinear Programming, Math. Progam., 106 (2006), 25–57.

[17]. Gleixner, A., Eifler, L., Gally, T., Gamrath, G., Gemander, P., Gottwald, R. L., Hendel, G., Hojny, C., Koch, T., Miltenberger, M., Muller, B., Pfetsch, M. E., Puchert, E., Rehfeldt, D., Schlosser, F., Serrano, F., Shinano, Y., Viernickel, J. M., Vigerske, S., Weninger, D., Witt, J. T., and Witzig, J. (2017), The SCIP optimization suite 5.0. Zuse Institute Berlin, ZIB-Report 17-61, December.

[18]. Burachik, R. S., Kaya, C. Y., Rizvi, M. M.: Algorithms for Generating Pareto Fronts of Multi-objective Integer and Mixed-Integer Programming Problems. arXiv:1903.07041v1. (2019).