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JORDAN DERIVATIONS ON LIE IDEALS OF σ-PRIME RINGS

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ABSTRACT

In this paper we prove that, if U is a σ-square closed Lie ideal of a 2-torsion free σ-prime ring R and $d: R(R$ is an additive mapping satisfying $d(u^2) = d(u)u + ud(u)$, for all $u \in U$ then $d(uv) = d(u)v + u\,d(v)$ holds for all $u, v \in U$.

Keywords: Lie ideal, σ-square closed Lie ideal, σ-prime ring, Jordan derivation, derivation.

1. Introduction

Throughout the paper, we consider R to be an associative ring with centre Z. $[a, b] = ab - ba$ which denotes the commutator of a and b, we will use the identities: $[ab, c] = [a, c]b + a\,[b, c]$ and $[a, bc] = [a, b]c + b[a, c]$ for all $a, b, c \in R$. An additive subgroup U of R is called a Lie ideal if $[U, R] \subseteq U$. An additive mapping $d: R (R$ is called a derivation if $d(ab) = d(a)b + ad(b)$ holds for all $a, b \in R$ and it is called a Jordan derivation if $d(a^2) = d(a)a + ad(a)$ holds for all $a \in R$. Clearly every derivation is a Jordan derivation but the converse is not true in general. A ring R is said to be a prime ring if $aRb = 0\ (a, b \in R)$ implies that $a = 0$ or $b = 0$. An additive mapping $f: R(R$ is called a generalized derivation with the associated derivation $d{:}R(R$ if $f(ab) = f(a)b + a\,d(b)$ holds for all $a, b \in R$; it is called a Jordan generalized derivation with the associated derivation d of R such that $f(a^2) = f(a)a + ad(a)$ holds for all $a \in R$. R. Awtar [1] proved that if $U \nsubseteq Z$ is a square closed Lie ideal of a 2-torsion free prime ring R and $d{:}R(R$ is an additive mapping such that $d(u^2) = d(u)u + ud(u)$, for all $u \in U$ then $d(uv) = d(u)v + u\,d(v)$ holds for all $u, v \in U$.

We need the following lemmas due to R. Awtar [1] for proving our result.

Lemma 1.1 If $U \nsubseteq Z$ is a Lie ideal of a ring R, then $d(uv + vu) = d(u)v + ud(v) + d(v)u + vd(u)$ holds for all $u, v \in U$.

Lemma 1.2 If $U \nsubseteq Z$ is a Lie ideal of a ring R, then $d(uvu) = d(u)vu + ud(v)u + uvd(u)$ holds for all $u, v \in U$.

Lemma 1.3 If $U \nsubseteq Z$ is a Lie ideal of a ring R, then $d(uvw + wvu) = d(u)vw + ud(v)w + uvd(w) + d(w)vu + wd(v)u + wvd(u)$ holds for all $u, v, w \in U$.

Lemma 1.4 If $U \not\subseteq Z$ is a Lie ideal of a ring *R*, then $u^v[u,v]=0$ holds for all $u,v \in U$, where $u^v = d(uv) - d(u)v - u\,d(v)$.

Lemma 1.5 If $U \not\subseteq Z$ is a Lie ideal of a ring *R*, then $[u,v]u^v = 0$ for all $u,v \in U$, where u^v is as in Lemma1.4

2. Jordan Derivations on Lie Ideals of σ-Prime Rings

Let *R* be a ring. A mapping $\sigma: R \to R$ is called an involution if $\sigma(a+b)=\sigma(a)+\sigma(b)$, $\sigma^2(a)=a$ and $\sigma(ab)=\sigma(b)\sigma(a)$ holds for all $a,b \in R$. A Lie ideal *U* of *R* is called a σ-Lie ideal if $\sigma(U)=U$ and it is called a σ-square closed Lie ideal if it is a σ-Lie ideal and for all $u \in U, u^2 \in U$. A ring *R* with involution σ is said to be a σ-prime ring if $aRb = aR\sigma(b) = \{0\}$ implies that *a*=0 or *b*=0. It is worthwhile to note that every prime ring having an involution σ is σ-prime but the converse is not true in general. As an example, let $T = R \times R^0$, where R^0 is an opposite ring of a prime ring *R* with involution $(x,y)=(y,x)$. Then T is not prime if $(0,a)T(a,0)=0$. But, *R* is σ-prime if we set $(a,b)T(x,y)=0$ and $(a,b)T\,(\sigma(x,y))=0$, then $aRx \times yRb = 0$ and $aRy \times xRb = 0$ and thus $aRx = yRb = aRy = xRb = 0$ by Oukhtite and Salhi [6]. We define the set $S_{a\sigma}(R) = \{x \in R: \sigma(x) = \pm x\}$ which are known as the set of symmetric and skew symmetric elements of *R*. Let *U* be a Lie ideal of *R*. We define $C_R(U) = \{r \in R: ru = ur, \forall\, u \in U\}$ which we shall call the centralizer of *U* with respect to *R*. Oukhtite and Salhi [12] worked on left derivation on σ-prime rings and proved that $U \subseteq Z$ or $d(U)=0$, where *U* is a nonzero σ-square closed Lie ideal of *R*. Oukhtite and Salhi [12] described additive mappings $d: R \to R$ such that $d(u^2) = 2ud(u)\ \forall u \in U$, where *U* is a nonzero σ-square closed Lie ideal of a 2-torsion free σ-prime ring *R* and prove that $d(uv) = ud(v) + vd(u)$ for all $u,v \in U$. Afterwords, Oukhtite, Salhi and Taoufiq [11] studied Jordan generalized derivations on σ-prime rings and proved that every Jordan generalized derivation on *U* of *R* is a generalized derivation on *U* of *R*, where *U* is a σ-square closed Lie ideal of a 2-torsion free σ-prime ring *R*. Some significant results on Lie ideals and generalized derivations in σ-prime rings have been obtained by M. S. Khan and M. A. Khan [5]. On the other hand, various remarkable characterizations of σ-prime rings on σ-square closed Lie ideals have been studied by many authors viz. M. R. Khan, D. Arora and M. A. Khan [4] ; Oukhtite and Salhi [7, 8, 9,10] and J. Bergun, I. N. Herstein and J. W. Kerr [2] and I. N. Herstein [3]. In this paper, we shall prove that if $d: R \to R$ is an additive mapping satisfying $d(u^2) = 2ud(u)\ \forall u \in U$, where *U* is a σ-square closed Lie ideal of a 2-torsion free σ-prime ring *R* then $d(uv) = ud(v) + vd(u)$ for all $u,v \in U$ and hence every Jordan derivations on a σ-prime ring *R* is a derivation on *R*. We begin with the following results.

Lemma 2.1 Let *R* be a 2-torsion free σ-prime ring and *U* be a σ-Lie ideal of *R*. Let $u \in U$ be any element such that $[u,[u,x]]=0$, for all $x \in R$, then $[u,x]=0$.

Proof: We have $[u,[u,x]]=0$ for all $x \in R$. Let $y \in R$, then $xy \in R$. Replacing *x* by *xy*, we have $[u,[u,xy]]=0$. So $0 = [u,[u,xy]] = [u,x[u,y] + [u,x]y]$

$$= [u, x[u, y]] + [u, [u, x]y]$$
$$= x[u, [u, y]] + [u, x][u, y] + [u, x][u, y] + [u, [u, x]]y$$
$$= 2[u, x][u, y].$$

Since R is 2-torsion free so $[u, x][u, \quad y] = 0$. For every $z \in R$ we have $zy \in R$. Putting zx for y, we have $[u, x][u, \quad zx] = 0.$ Therefore, $0 = [u, x](z[u, x] + [u, z]x)$

$$= [u, x]z[u, x] + [u, x][u, z]x$$
$$= [u, x]z[u, x].$$

Therefore, $[u, x]R[u, x] = 0.$ Since $((U) = U,$ we have $((u) = u$, for all $u \in U$. Let $x \in S_{a\sigma}(R).$ Then $((x) = \pm x.$ If $((u) = u$ and $((x) = -x,$ then

Hence $[u, x]R[u, x] = [u, x]R(([u, x] = 0.$ By the σ-primeness of R, we get $[u, x] = 0.$

Lemma 2.2 Let R be a 2-torsion free σ-prime ring and $U \neq 0$ be a σ-Lie ideal and a σ- subring of R. Then either $U \subseteq Z$ or U contains a nonzero σ-ideal of R.

Proof: First we assume that, U as a σ-ring is not commutative. Then for some $u, v \in U, [u, v] \neq 0$ and $[u, v] \in U.$ Therefore the ideal J of R generated by $[u, v]$ is nonzero, $J \subseteq U$ and $((J) = J.$ On the other hand, let us assume that U is commutative. Then for every $u \in U, [u, [u, x]] = 0$ for all $x \in R.$ Hence by Lemma 2.1, $[u, x] = 0.$ This shows that $U \subseteq Z.$

Lemma 2.3 If $U \nsubseteq Z$ is a σ-Lie ideal of a σ-prime ring R, then $C_R(U) = Z.$

Proof: $C_R(U)$ is both a σ-subring and a σ-Lie ideal of R and $C_R(U)$ contains no nonzero σ-ideal of R. In view of Lemma 2.2, $C_R(U) \subseteq Z.$ Therefore, $C_R(U) = Z.$

Lemma 2.4 If U is a σ-Lie ideal of a σ-prime ring R and $a \in R$. If $[a, [U, U]] = 0$ then $[a, U] = 0,$ that is, $C_R([U, U]) = C_R(U).$

Proof: If $[U, U] \nsubseteq Z,$ then by Lemma 2.3, $a \in Z,$ so a centralizes U. On the other hand, let $[U, U] \subseteq Z,$ then we have $[u, [u, x]] = 0$ for $u \in U$ and $x \in R$. In view of Lemma 2.1, $[u, x] = 0.$ This yields that $U \subseteq Z.$ For both the cases we have seen that $a \in C_R(U).$ This gives that $C_R([U, U]) = C_R(U).$

Lemma 2.5 Let $U \nsubseteq Z$ be a σ-square closed Lie ideal of a 2-torsion free σ-prime ring R and $d : R(R$ be an additive mapping satisfying $d(u^2) = d(u)u + u\, d(u),$ for all $u \in U$. If $u^v = d(uv) \mathrm{I}\, d(u)v - u\, d(v),$ for all $u, v \in U$ then $u^v w[u, v] = 0,$ for all $w \in U$.

Proof: In view of Lemmas 1.4 and 1.5, we have $[u^v, \quad [u, v]] = u^v[u, v] - [u, v]u^v = 0$. This yields that $u^v \in C_R([U, U]) = C_R(U),$ by Lemma 2.4. Hence for every $w \in U$, we have $u^v w[u, v] = 0.$

Lemma 2.6 ([7], Lemma 2.2) Let $U \nsubseteq Z$ be a σ-Lie ideal of a 2-torsion free σ-prime ring *R* and $a, b \in R$ such that $aUb = aU((b) = 0,$ then $a = 0$ or $b = 0.$

Theorem 2.7 Let *U* be a σ-square closed Lie ideal of a 2-torsion free σ-prime ring *R* and $d{:}R(\ R$ be an additive mapping satisfying $d(u^2) = d(u)u + u\,d(u),$ for all $u \in U$, then $d(uv) = d(u)v + ud(v)$ holds for all $u, v \in U$.

Proof: If *U* is a non-commutative Lie ideal of *R* then $U \nsubseteq Z.$ By Lemma 2.5, we have $a^b w[a,b] = 0$ for all $a, b, w \in U$. Let us assume that $a, b \in U \cap S_{a\sigma}(R).$ Since $((U) = (U),$ we have $([a,b] = [a,b]$ as $[a,b] \in U$. If $((b) = \mathbf{I}b$ and $((a) = a$, then Also, if $((b) = b$ and $((a) = \mathbf{I}a,$ then $([a,b] = [a,b].$ Therefore, we have $a^{\dagger}b\,w[a,b] = a^{\dagger}b\,w\,([a,b] = 0.$ By applying the Lemma 2.6 in the above relation, we obtain that $a^b = 0$ or $[a,b] = 0$ for all $a, b \in U \cap S_{a\sigma}(R).$ Let $I_a = \{b \in U: a^b = 0\}$ and $J_a = \{b \in U: [a,b] = 0\}.$ Then I_a and J_a are additive subgroups of U such that $I_a \bigcup J_a = U$. Then by Brauer's trick $I_a = U$ or $J_a = U$. Using the similar argument, we have $U = \{a \in U: U = I_a\}$ or $U = \{a \in U: U = J_a\}.$ If $U = \{a \in U: U = J_a\}$ then $[a,b] = 0$, which yields that $U \subseteq Z$, by Lemma 2.2. Which is a contradiction to the fact that $U \nsubseteq Z.$ So we have $U = \{a \in U: U = I_a\}$ and hence $a^b = 0$ for all $a, b \in U \cap S_{a\sigma}(R).$ This implies

$$d(ab) = d(a)b + ad(b), \forall\ a, b(U \cap S_1 a\sigma\ (R) \ldots\ldots\ldots \quad \textbf{(1)}$$

Now let $u, v \in U.$ If we define $u_1 1 = u + ((u),\quad u_1 2 = u\,\mathbf{I}\,((u), v_1 1 = v + ((v), v_1 2 = v\,\mathbf{I}\,((v).$ Then $u_1, u_2, v_1, v_2 \in U \cap S_{a\sigma}(R)$ and we have $2u = u_1 + u_2, 2v = v_1 + v_2.$ Therefore, in view of (1), we obtain

$$d(2u2v) = d(u_1 v_1 + u_1 v_2 + u_2 v_1 + u_2 v_2)$$

$$= d(u_1)v_1 + u_1 d(v_1) + d(u_1)v_2 + u_1 d(v_2) + d(u_2)v_1 + u_2 d(v_1) + d(u_2)v_2 + u_2 d(v_2)$$

$$= \big(d(u_1) + d(u_2)\big)v_1 + (u_1 + u_2)d(v_1) + \big(d(u_1) + d(u_2)\big)v_2 + (u_1 + u_2)d(v_2)$$

$$= d(u_1 + u_2)v_1 + 2ud(v_1) + d(u_1 + u_2)v_2 + 2ud(v_2) = d(2u)v_1 + 2u\,d(v_1) + d(2u)v_2 + 2ud(v_2) = 2c$$

$$= 2d(u)2v + 2u\,d(2v)$$

$$= 4d(u)v + 4u\,d(v).$$

Thus $4d(uv) = 4\big(d(u)v + u\,d(v)\big).$ Since *R* is 2-torsion free, we obtain

$d(uv) = d(u)v + u\,d(v).$ If *U* is a commutative σ-Lie ideal of *R*, then by Lemma 2.2, $U \subseteq Z.$ Therefore, by using 2-torsion freeness of *R* and in view of the Lemma 1.1, we have $d(uv) = d(u)v + ud(v).$

In view of above theorem, we obtain the following corollary.

Corollary 2.8 Let *R* be a 2-torsion free σ-prime ring. Then every Jordan derivations on *R* is a derivation on *R*.

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