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ON SOME VALUES OF THE SANDOR-SMARANDACHE FUNCTION

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ABSTRACT

Sandor [1] posed a new function, denoted by *SS(n)*, and is defined as follows : $SS(n) = \max\left\{m : 1 \le m \le n-1,\ n \text{ divides} \binom{n}{m}\right\}, n \ge 2$ $\binom{n}{m} = \frac{n!}{m!(n-m)!}$ being the binomial coefficients. This paper finds *SS(n)* for some particular cases of n.

Keywords : Sandor-Smarandache function, Binomial coefficient, Diophantine equation

1. Introduction

Let $C(n, m)$ be the binomial coefficient, defined as follows :

$$C(n, m) = \binom{n}{m} = \frac{n!}{m!\,(n-m)!},\ 0 \le m \le n.$$

Then, the Sandor-Smarandache function, denoted by $SS(n)$, is defined as follows :

$$SS(n) = \max\left\{m : 1 \le m \le n-1,\ n \text{ divides } \binom{n}{m}\right\},\ n \ge 3,$$

with

$$SS(1) = 1,\ SS(2) = 1,\ SS(3) = 1,\ SS(4) = 1,\ SS(6) = 1.$$

Throughout this paper, we use the following formula for $C(n, m)$:

$$C(n, m) = \frac{n(n-1)(n-2)\ldots(n-m+1)}{m!},\ 0 \le m \le n.$$

Sandor [1] proved the result below.

Lemma 1.1 : $SS(n) = n-2$ for any odd integer $n\ (\ge 3)$.

Corollary 1.1 : For any prime $p \ge 3$, $SS(p) = p-2$, and in general,

$$SS(p_1 p_2 \ldots p_s) = p_1 p_2 \ldots p_s - 2 \text{ for any odd primes } p_1, p_2, \ldots, p_s.$$

As has been pointed out by Sandor [1], to find $SS(n)$, the case of even n is more involving. This paper considers this case in the next section. We first derive $SS(n)$ when $n = p + 1$, p being an odd prime. Next, we confine our attention to the prime p of the four forms, $p = 5v + 1$, $p = 5v + 2$, $p = 5v + 3$ and $p = 5v + 4$ (for some integer $v \geq 1$).

2. Main Results

In a recent book, Majumdar [2] derived the expressions of $SS(n)$ for some particular cases. In this section, we derive more to supplement the results found in [2]. We concentrate on the functions $SS(p + 1)$, where p is an odd prime of the forms $p = 5v + 1, 5v + 2, 5v + 3, 5v + 4$.

We first prove the following simple result.

Lemma 2.1 : Let p (≥ 5) be an odd prime. Then, $SS(p + 1) = p - 2$

if and only if $p + 1$ is not a multiple of 3.

Proof : We consider

$$\frac{(p+1)p(p-1)}{3!}.$$

If 3 does not divide $p + 1$, then 3 must divide $p - 1$, and hence, 6 divides $p - 1$.

Conversely, if $SS(p + 1) = p - 2$, then 3 must divide $p - 1$, and consequently, $p + 1$ is not divisible by 3.

Applying Lemma 2.1, we get the following expressions :

$$SS(8) = 5, SS(14) = 11, SS(20) = 17, SS(32) = 29, SS(38) = 35, SS(44) = 41.$$

Corollary 2.1 : Let p (≥ 5) be an odd prime such that 3 divides $p + 1$. Then,

$$SS(p + 1) \geq p - 3.$$

Proof : follows immediately by virtue of Lemma 2.1.

Lemma 2.2 – Lemma 2.7 below deal with the case when p is a prime of the form $p = 5v + 1$.

Lemma 2.2 : Let p be a prime of the form $p = 5v + 1$ for some integer $v \geq 1$. Then,

$$SS(p + 1) = p - 3,$$

if $v = 8(3s + 1)$ for some integer $s \geq 0$.

Proof : With $p = 5v + 1$, the following expression

$$(p+1)p\frac{(p-1)(p-2)}{4!}$$

takes the form

$$5(p+1)p\frac{v(5v-1)}{2\times3\times4}.$$

We consider the case when 8 divides v and 3 divides $5v-1$, so that

$v = 8x$, $5v = 3y + 1$ for some integers $x \geq 1$, $y \geq 1$.

The solution of the second Diophantine equation is $v = 3a + 2$, $a \geq 0$. This, when combined with the first equation, gives

$8x = 3a + 2$,

whose solution is $x = 3s + 1$, $s \geq 0$. Hence, finally

$v = 8x = 8(3s + 1)$.

Observe that, in Lemma 2.2, $p = 5v + 1 = 3(8s + 3)$ is a multiple of 3, a result consistent with Corollary 2.1.

Using Lemma 2.2, we get the functions below :

$SS(42) = 38$, $SS(282) = 278$, $SS(402) = 398$, $SS(522) = 518$, $SS(642) = 638$.

Lemma 2.3 : Let p be a prime of the form $p = 5v + 1$ for some integer $v \geq 1$. Then,

$SS(p + 1) = p - 4$,

if $v = 2(6s + 1)$ for some integer $s \geq 0$.

Proof : With $p = 5v + 1$, we have

$$(p+1)p\frac{(p-1)(p-2)(p-3)}{5!} = (p+1)p\frac{v(5v-1)(5v-2)}{2\times3\times4}.$$

Now, we consider the case when 3 divides $5v-1$ and 4 divides $5v-2$. Then,

$5v = 3x + 1$, $5v = 4y + 2$ for some integers $x \geq 1$, $y \geq 1$.

The solutions of these two Diophantine equations are

$v = 3a + 2 = 4b + 2$; $a \geq 0$, $b \geq 0$ being any integers.

This shows that $a = 4s$, $s \geq 1$. Therefore,

$v = 3a + 2 = 2(6s + 1)$,

which is the desired condition.

Lemma 2.3 gives the following functions :

$SS(12) = 7$, $SS(72) = 65$, $SS(132) = 125$, $SS(192) = 185$, $SS(252) = 247$.

The first example shows that Lemma 2.3 is valid for $s = 0$ as well.

From the proof of Lemma 2.1, it may be deduced that, if $p = 5v + 1$, then $SS(p + 1) = p - 2$ if and only if $v = 6s$ $(s \geq 1)$. However, in other cases, there might be more than one solution, as the two lemmas below illustrate.

Lemma 2.4 : Let p be a prime of the form $p = 5v + 1$ with $v = 4(3s + 2)$, $s \geq 0$. Then,

$$SS(p + 1) = \begin{cases} p-3, & \text{if } s \text{ is even} \\ p-4, & \text{if } s \text{ is odd} \end{cases}$$

Proof : Consider the expression

$$(p+1)p\frac{(p-1)(p-2)(p-3)}{5!} = (p+1)p\frac{v(5v-1)(5v-2)}{2\times3\times4}.$$

Now, let s be even, say, $s = 2r$ for some integer $r \geq 1$. Then,

$$v = 4(6r + 2) = 8(3r + 1),\ 5v - 1 = 3(40r + 13).$$

Then, clearly $(p-1)(p-2) = 5v(5v-1)$ is divisible by $4!$, and hence, $SS(p + 1) = p - 3$.

Next, let s be odd of the form $s = 2t + 1$ for some integer $t \geq 1$. Then,

$$v = 2(12t + 7),\ 5v - 1 = 3(40t + 13),\ 5v - 2 = 4(30t + 17).$$

In this case, $(p-1)(p-2)$ is not divisible by $4!$, but $(p-1)(p-2)(p-3)$ is divisible by $4!$.

From Lemma 2.4, corresponding to $s = 0$, we get the prime $p = 41$ with $SS(42) = 38$; when $s = 1$, we get the prime $p = 101$ which gives $SS(102) = 97$. Again, with $s = 2$, we get the prime $p = 161$ with $SS(162) = 158$. The next prime in the sequence is $p = 281$ (corresponding to $s = 4$) with $SS(282) = 278$.

Lemma 2.5 : Let p be a prime of the form $p = 5v + 1$ with $v = 2(3s + 1)$, $s \geq 0$. Then,

$$SS(p + 1) = \begin{cases} p-3, & \text{if } s = 4t + 1,\ t \geq 0 \\ p-4, & \text{otherwise} \end{cases}$$

Proof : We start with

$$(p+1)p\frac{(p-1)(p-2)(p-3)}{5!} = (p+1)p\frac{v(5v-1)(5v-2)}{2\times3\times4}.$$

With $v = 2(3s + 1)$, $(p-1)(p-2) = 30(3s + 1)(10s + 3)$.

When $s = 4t + 1$, then $3s + 1 = 4(3t + 1)$, so that $4!$ divides $(p-1)(p-2)$. Thus, in this case,

$$SS(p + 1) = p - 3.$$

Otherwise, $4!$ does not divide $(p-1)(p-2)$, but $(p-1)(p-2)(p-3) = 60(3s + 1)(10s + 3)(15s + 4)$

is divisible by $5!$, so that $SS(p + 1) = p - 4$.

From Lemma 2.5, corresponding to $s = 0$, we get the prime $p = 11$ with $SS(12) = 7$; $s = 1$ gives the prime $p = 41$, which is of the form $4t + 1$, so that $SS(42) = 38$. Corresponding to $s = 3$, we get the prime $p = 101$ with $SS(102) = 97$. Continuing, we get successively the functions $SS(132) = 125$,

$SS(192) = 187$, $SS(252) = 247$ and $SS(282) = 278$.

Lemma 2.6 : Let p be a prime of the form $p = 5v + 1$ with $v = 2(9s + 10)$, $s \geq 0$. Then,

$$SS(p + 1) = \begin{cases} p-3, & \text{if } s \neq 4t + 2,\ t \geq 0 \\ p-4, & \text{otherwise} \end{cases}$$

Proof : Consider the expression below :

$$(p+1)p\frac{(p-1)(p-2)(p-3)}{5!} = (p+1)p\frac{v(5v-1)(5v-2)}{2\times3\times4}.$$

If $s = 4t + 2$, then

$$9s + 10 = 4(9t + 7).$$

Therefore,

$$(p-1)(p-2) = 90(9s + 10)(10s + 11),$$

which is divisible by $4!$. Thus, in this case,

$$SS(p + 1) = p-3.$$

Otherwise, $(p-1)(p-2)$ is not divisible by $4!$, but $5!$ divides

$$(p-1)(p-2)(p-3) = 180(9s + 10)(10s + 11)(45s + 49)$$

so that $SS(p + 1) = p-4$.

Some functions, obtained from Lemma 2.6, are listed below :

$SS(102) = 97$, $SS(192) = 187$, $SS(282) = 278$, $SS(462) = 457$, $SS(642) = 638$,

$SS(822) = 817$, $SS(912) = 907$, $SS(1092) = 1087$, $SS(1182) = 1177$, $SS(1362) = 1358$.

Lemma 2.7 : Let p be a prime of the form $p = 5v + 1$ for some integer $v \geq 1$. Then,

$$p-2 \leq SS(p + 1) \leq p-4.$$

Proof : We prove the lemma by showing that $SS(p + 1) \neq p - 5$. So, we consider the expression below :

$$(p+1)p\frac{(p-1)(p-2)(p-3)(p-4)}{6!} = (p+1)p\left[\frac{v(5v-1)(5v-2)(5v-3)}{2\times3\times4\times6}\right].$$

Now, we find the condition under which the term inside the square bracket on the right is an integer. To do so, first note that, by Corollary 2.1, (since 3 divides $p + 1$), 9 must divide $5v-1$. Then, since $5v-3$ must be odd, we see that it remains dormant, and consequently, in such a case, we must have $SS(p + 1) \leq p-4$.

The next three lemmas deal with the case when p is a prime of the form $p = 5v + 2$.

Lemma 2.8 : Let p be a prime of the form $p = 5v + 2$ for some integer $v \geq 1$. Then,

$SS(p+1) = p-3,$

if $v = 3(8s+1)$ for some integer $s \geq 0$.

Proof : With $p = 5v+2$, we get

$$(p+1)p\frac{(p-1)(p-2)}{4!} = 5(p+1)p\frac{v(5v+1)}{2\times3\times4}.$$

Now, consider the case when 8 divides $5v+1$ while 3 divides v, so that

$5v = 8x-1$, $v = 3y$ for some integers $x \geq 1$, $y \geq 1$.

The solution of the first equation is

$v = 8a+3$, $a \geq 0$.

We are then lead to the equation

$8a = 3y-3$,

with the solution $a = 3(s+1)$, $s \geq 0$. Plugging in this expression in $v = 8a+3$, we get the desired condition.

The following functions are obtained from Lemma 2.8 :

$SS(18) = 14$, $SS(138) = 134$, $SS(258) = 254$, $SS(618) = 614$, $SS(858) = 854$.

Lemma 2.9 : Let p be a prime of the form $p = 5v+2$ for some integer $v \geq 1$. Then,

$SS(p+1) = p-4,$

if $v = 3(4s+3)$, $s \geq 0$.

Proof : We start with

$$(p+1)p\frac{(p-1)(p-2)(p-3)}{5!} = (p+1)p\frac{v(5v+1)(5v-1)}{2\times3\times4}.$$

Now, we consider the case when 4 divides $5v-1$ and 3 divides v. Then,

$5v = 4x+1$, $v = 3y$ for some integers $x \geq 1$, $y \geq 1$.

The first equation has the solution

$v = 4a+1$, $a \geq 0$.

We are then faced with the Diophantine equation

$v = 3y = 4a+1$,

whose solution is

$a = 3s+2$, $s \geq 0$.

After simplification, we get the condition desired.

From Lemma 2.9, we get the functions below.

$$SS(48) = 43, SS(108) = 103, SS(158) = 153, SS(228) = 223, SS(348) = 343.$$

Lemma 2.10 : Let p be a prime of the form $p = 5v + 2$ with $v = 3(2s + 1), s \geq 0$. Then,

$$SS(p + 1) = \begin{cases} p - 3, & \text{if } s \neq 4t,\ t \geq 0 \\ p - 4, & \text{otherwise} \end{cases}$$

Proof : Consider the expression below :

$$(p+1)p\frac{(p-1)(p-2)(p-3)}{5!} = (p+1)p\frac{v(5v+1)(5v-1)}{2\times3\times4}.$$

Now, we consider the case when 2 divides $5v + 1$ and 3 divides v. Then,

$$5v = 2x - 1, v = 3y \text{ for some integers } x \geq 1, y \geq 1.$$

The first Diophantine equation has the solution

$$v = 2a + 1, a \geq 0,$$

which, combined with the second equation, leads to

$$3y = 2a + 1,$$

which gives

$$y = 2s + 1, s \geq 0.$$

And finally, we get $v = 3(2s + 1)$.

Now, since

$$5v + 1 = 2(15s + 8),$$

it follows that

$$SS(p + 1) = p - 4, \text{ if } 4 \text{ divides } 15s + 8;$$

otherwise, $SS(p + 1) = p - 5$.

Now, noting that 4 divides $15s + 8$ if and only if $s = 4t, t \geq 1$, the lemma is established.

Some functions, obtained from Lemma 2.10, are

$$SS(18) = 14, SS(48) = 43, SS(108) = 103, SS(138) = 134, SS(168) = 163,$$

$$SS(198) = 193, SS(228) = 223, SS(258) = 254.$$

Lemma 2.11 – Lemma 2.14 consider the case when the prime p is of the form $p = 5v + 3$.

Lemma 2.11 : Let p be a prime of the form $p = 5v + 3$, where $v = 2(12s + 11), s \geq 0$. Then,

$SS(p + 1) = p - 3.$

Proof : Letting $p = 5v + 3$ in

$$(p + 1)p\frac{(p - 1)(p - 2)}{4!},$$

we get

$$(p + 1)p\left[\frac{(5v + 2)(5v + 1)}{2\times3\times4}\right].$$

Now, in order that the above number is an integer, 8 must divide $5v + 2$, and 3 must divide $5v + 1$. This leads to the following two Diophantine equations

$5v = 8x - 2$, $5v = 3y - 1$ for some integers $x\ (\geq 1)$ and $y\ (\geq 1)$,

with solutions

$v = 8a + 6$, $v = 3b + 1$ ($a \geq 1$ and $b \geq 1$ being integers),

respectively. Now, combining together, the resulting equation is

$8a = 3b - 5,$

whose solution is

$a = 3s + 2,\ s \geq 0.$

Hence,

$v = 8(3s + 2) + 6 = 2(12s + 11),$

which is the desired expression we were looking for.

Applying Lemma 2.11, we get the expressions below :

$SS(114) = 110,\ SS(234) = 230,\ SS(594) = 590.$

Lemma 2.12 **:** Let p be a prime of the form $p = 5v + 3$, $v = 4(3s + 1)$ for some integer $s \geq 0$. Then,

$SS(p + 1) = p - 4.$

Proof : We start with

$$(p + 1)p\frac{(p - 1)(p - 2)(p - 3)}{5!} = (p + 1)p\left[\frac{(5v + 2)(5v + 1)v}{2\times3\times4}\right].$$

We consider the case when 3 divides $5v + 1$, 2 divides $5v + 2$ and v itself is a multiple of 4. Then, we have

$5v = 3x - 1$, $5v = 2y - 2$, $v = 4z$ for some integers $x\ (\geq 1)$, $y\ (\geq 1)$ and $z\ (\geq 1)$.

The first Diophantine equation gives the solution

$v = 3a + 1,\ a \geq 1.$

This, together with the condition $v = 4z$, requires that

$3a = 4z - 1,$

whose solution is

$a = 4s + 1,\ s \geq 1.$

Therefore,

$v = 3a + 1 = 3(4s + 1) + 1 = 4(3s + 1).$

Applying Lemma 2.12, we get the functions below :

$SS(24) = 19,\ SS(84) = 79,\ SS(264) = 259,\ SS(384) = 379.$

Lemma 2.13 : Let p be a prime of the form $p = 5v + 3$, $v = 8(3s + 2)$ for some integer $s \geq 1$. Then,

$SS(p + 1) = p - 4.$

Proof : We start with the following simplified form :

$$(p+1)p\frac{(p-1)(p-2)(p-3)}{5!} = (p+1)p\left[\frac{(5u+2)(5u+1)u}{2\times 3\times 4}\right].$$

Now, we consider the case when 3 divides $5v + 1$ and 8 divides v. Then,

$5v = 3x - 1$, $v = 8y$ for some integers $x\ (\geq 1)$ and $y\ (\geq 1)$.

The first equation has the solution $v = 3a + 1$, $a\ (\geq 1)$, and hence, we have to consider

$3a = 8y - 1,$

whose solution is $a = 8s + 5,\ s \geq 1$.

Therefore,

$v = 3(8s + 5) + 1 = 8(3s + 2),$

which we intended to establish.

Application of Lemma 2.13 gives the functions :

$SS(84) = 79,\ SS(444) = 439.$

Lemma 2.14 : Let p be a prime of the form $p = 5v + 3$, where $v = 2(3s + 2)$ for some integer $s \geq 1$ such that $s \neq 4t + 3$ for any $t \geq 0$. Then,

$SS(p + 1) = p - 4.$

Proof : We start with

$$(p+1)p\frac{(p-1)(p-2)(p-3)}{5!}=(p+1)p\left[\frac{(5v+2)(5v+1)v}{2\times3\times4}\right].$$

We consider the case when 3 divides $5v+1$ and 2 divides v. Then,

$5v=3x-1$, $v=2y$ for some integers $x\ (\geq 1)$ and $y\ (\geq 1)$.

The solution of the first equation is

$v=3a+1,\ a\geq 1.$

Then, the combined Diophantine equation is

$2y=3a+1,$

with the solution

$y=3s+2,\ s\geq 0.$

Therefore,

$v=3(8s+5)+1=8(3s+2).$

Note that, $5v+2=2(15s+11)$, and hence, to complete the proof, we have to guarantee that 4 does not divide $15s+11$. To do so, we consider the equation $15s=4b-11$, which has the solution

$s=4t+3,\ t\geq 0.$

Thus, if $s\neq 4t+3$ for any $t\geq 0$, then $SS(p+1)=p-4$.

Hence, the proof of the lemma is complete.

Lemma 2.14 gives the following functions

$SS(24)=19,\ SS(54)=49,\ SS(84)=79,\ SS(174)=169,\ SS(264)=259.$

It may be mentioned here that, if $s=4t+3\ (t\geq 0)$ in Lemma 2.14, then by Lemma 2.11,

$SS(p+1)=p-3.$

Finally, we have the following lemma, dealing with the case when p is a prime of the form

$p=5v+4.$

Lemma 2.15 : Let p be a prime of the form $p=5v+4$, $v=24s+17$ for some $s\geq 0$. Then,

$SS(p+1)=p-3.$

Proof : With $p=5v+4$, we get

$$(p+1)p\frac{(p-1)(p-2)}{4!}=(p+1)p\left[\frac{(5v+3)(5v+2)}{2\times3\times4}\right].$$

We consider the case when 8 divides $5v+3$ and 3 divides $5v+2$. Thus,

$$5v + 3 = 8x,\ 5v + 2 = 3y \text{ for some integers } x\ (\geq 1) \text{ and } y\ (\geq 1), \tag{1}$$

that is, $5v = 8x - 3$, $5v = 3y - 2$.

The solutions of the above two equations are

$$v = 8a + 1,\ v = 3b + 2;\ a \geq 1 \text{ and } b \geq 1 \text{ being integers.}$$

Now, combining together the above two Diophantine equations, we get the equation

$$8a = 3b + 1,$$

with the solution

$$a = 3s + 2,\ s \geq 0.$$

Thus,

$$v = 8(3s + 2) + 1 = 24s + 17,$$

which is the condition desired.

Some of the functions found from Lemma 2.15 are given below.

$$SS(90) = 86,\ SS(450) = 446,\ SS(570) = 566,\ SS(810) = 806.$$

Remark 2.1 : In the proof of Lemma 2.15, writing the two equations in (1) in the form

$$5v + 3 = 8x = 3y + 1,$$

we get the solution

$$x = 3t + 2,\ t \geq 0.$$

Then,

$$p = 5v + 4 = 8x + 1 = 8(3t + 2) + 1 = 24t + 17.$$

Having the solution in the above form, we get the functions below :

$$SS(18) = 14,\ SS(42) = 38,\ SS(90) = 86,\ SS(114) = 110,\ SS(138) = 134,$$

$$SS(234) = 230,\ SS(258) = 254,\ SS(282) = 278,\ SS(354) = 350.$$

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