GANIT *J. Bangladesh Math. Soc.* (ISSN 1606-3694) **39** (2019) 1-14
DOI: https://doi.org/10.3329/ganit.v39i0.44158

ON THE STAR PUZZLE

A.A.K. Majumdar
Ritsumeikan Asia-Pacific University, Beppu 874-8577, Japan
E-mail : aakmajumdar@gmail.com

Received: 23-04-2018 Accepted: 24-03-2019

ABSTRACT

In the *star puzzle,* there are four pegs, the usual three pegs, S, P and D, and a fourth one at 0. Starting with a tower of n discs on the peg P, the objective is to transfer it to the peg D, in minimum number of moves, under the conditions of the classical Tower of Hanoi problem and the additional condition that all disc movements are either to or from the fourth peg. Denoting by *MS(n)* the minimum number of moves required to solve this variant, *MS(n)* satisfies the recurrence relation $MS(n) = \min_{1 \le k \le n-1}\{2MS(n-k)+3^k-1\}, n \ge 2;\ MS(0)=0$. This paper studies rigorously and extensively the above recurrence relation, and gives a solution of it.

Keywords : Star puzzle, Three-in-a-row puzzle, Recurrence relation

1. Introduction

The *star puzzle*, due to Stockmeyer [6], is as follows : There are four pegs, the usual three pegs S, P and D, and a fourth one at 0. Initially, the *n* discs, d_1, d_2, …, d_n, rest on the *source peg*, S, in a tower (with the smallest disc at the top, the second smallest above it, and so on, with largest disc at the bottom), as shown in the figure below. The objective is to transfer the tower from S to the *destination peg*, D, in minimum number of moves, (using the *auxiliary peg* P), under the conditions that (1) each move can shift only the topmost disc from one peg to another, (2) no disc can be placed on top of a smaller one, and (3) each disc movement is either to or from 0.

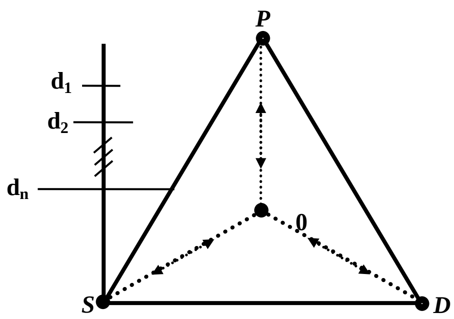

Fig. 1.1 : *The Star Puzzle*

Let $MS(n)$ be the minimum number of moves required to solve the star puzzle with n (≥ 1) discs. To find the recurrence relation satisfied by $MS(n)$, we consider the following scheme to transfer the tower of n discs from the source peg, S, to the destination peg, D.

Step 1 : move the topmost tower of $n-k$ discs from the peg S to the peg P, using the available four pegs, in (minimum) $MS(n-k)$ moves.

Step 2 : shift the k discs (resting on the peg S) to the peg D. Clearly, only three pegs are available.

In this step, the three pegs S, 0 and D (in this order) may be treated as being arranged in a row, forming the *three-in-a-row puzzle* with k discs. The minimum number of moves involved is 3^k-1.

Step 3 : transfer the tower of $n-k$ discs from the peg P to the peg D, completing the tower on the peg D, in (minimum) $MS(n-k)$ moves.

Thus, the total number of moves required is $FS(n,k) \equiv 2\,MS(n-k) + 3^k - 1$, where k ($1 \leq k \leq n-1$) is such that $FS(n, k)$ is minimized. Therefore, $MS(n)$ satisfies the following dynamic programming equation

$$MS(n) = \min_{1 \leq k \leq n-1}\{2MS(n-k) + 3^k - 1\},\ n \geq 2;\ MS(0) = 0 \tag{1.1}$$

with

$$MS(0) = 0,\ MS(1) = 2, \tag{1.2}$$

with the convention that $MS(1)$ is attained at the point $k = 1$.

Step 2 of the scheme involves a three-in-a-row puzzle, and we refer to Scorer, Grundy and Smith [5] and Majumdar [1] for details on the three-in-a-row puzzle.

Since $3^k - 1$ is even for all $k \geq 1$, we see that the term inside the curly brackets on the right-hand side of (1.1) is even. Thus, $MS(n)$ is even for all $n \geq 1$. Let

$$MSM(n) = \frac{MS(n)}{2} \tag{1.3}$$

Then, we have the following result.

Lemma 1.1 : $MS(n)$ is attained at $k = K$ if and only if $MSM(n)$ is attained at $k = K$.

Proof : is evident from the defining equation (1.3).

It may be noted here that, the recurrence relation satisfied by $MSM(n)$ is

$$MSM(n) = \min_{1 \leq k \leq n-1}\left\{2MSM(n-k) + \frac{1}{2}(3^k - 1)\right\},\ n \geq 2 \tag{1.4}$$

with

$$MSM(0) = 0,\ MSM(1) = 1. \tag{1.5}$$

Let

$$a_n = MSM(n) - MSM(n-1) \text{ for all } n \geq 1. \tag{1.6}$$

Let the sequence $\{b_n\}_{n=1}^{\infty}$ be defined as follows :

$$b_n = 2^i 3^m,\ i \geq 0,\ m \geq 0, \tag{1.7}$$

arranged in increasing order. In this paper, we give a rigorous proof of the result that

$$a_n = b_n \text{ for all } n \geq 1,$$

by showing that a_n and b_n satisfy the same recurrence relation. This is done in Section 3. In Section 2, we give some preliminary results related to $MSM(n)$.

2. Some Preliminary Results

Some local-value relationships satisfied by $MS(n)$ have been derived in Majumdar [3]. By Lemma 1.1, they may be adapted to $MSM(n)$, and are given below.

Lemma 2.1 : For any integer $n \geq 1$,

(a) $0 < MSM(n+1) - MSM(n) < MSM(n+2) - MSM(n+1)$

$$\leq 2\{MSM(n+1) - MSM(n)\},$$

(b) $MSM(n)$ is attained at a unique value of k.

Corollary 2.1 : For any integers $m \geq 1$, $n \geq 1$,

$$MSM(n+1) - MSM(n) = MSM(m+1) - MSM(m) \tag{1}$$

if and only if $m = n$.

Proof : The proof of the "if" part is trivial. To prove the "only if" part, let the equality (1) hold true for some integers m and n. Now, if $m \neq n$, then either $m > n$ or $n > m$. In either case, part (a) of Lemma 2.1 is violated.

Lemma 2.2 : Let $MSM(n)$ be attained at the point $k = k_1$ and $MSM(n+1)$ be attained at the point $k = k_2$. Then, $k_1 \leq k_2 \leq k_1 + 1$.

Lemma 2.2 above states that, if $MSM(n)$ is attained at the point $k = K$, then $MSM(n+1)$ is attained either at

$$k = K \text{ or at } k = K + 1.$$

Corollary 2.2 : Let $MSM(n)$ be attained at $k = K$ and $MSM(n+1)$ be attained at $k = K + 1$. Then, $MSM(n+2)$ is attained at $k = K + 1$.

Lemma 2.3 : Let $MSM(n) - MSM(n-1)$ be of the form 3^m for some integers $n \geq 1$, $m \geq 0$. Let $MS(n)$ be attained at $k = K$. Then, $MS(n-1)$ is attained at $k = K - 1$, and $MSM(n+1)$ is attained at $k = K$.

Lemma 2.4 : Let, for some $N \geq 1$, $MSM(N-1)$ be attained at $k = K$ and $MSM(N)$ be attained at $k = K + 1$, so that

$$MSM(N) - MSM(N-1) = 3^K. \tag{2.1}$$

Then, there is an integer $M \geq 1$ such that

$$MSM(N + M + 1) - MSM(N + M) = 3^{K+1}, \tag{2.2}$$

with $M = 1$ if and only if $N = 1$ (so that $K = 0$).

Moreover, $M > K$.

Proof : The first part of the lemma has been established in Majumdar [3]. It then remains to show that $M > K$.

Now, for any integer L with $N < L < M$, $MSM(N + L + 1)$ and $MSM(N + L)$ both are attained at the point $k = K + 1$, so that

$$MSM(N + L + 1) - MSM(N + L)$$

$$=2[MSM(N + L - K) - MSM(N + L - K - 1)].$$

Choosing $L = K$, we get

$$MSM(N + K + 1) - MSM(N + K)$$

$$= 2[MSM(N) - MSM(N-1)] = 2.3^K, \tag{2.3}$$

where, in the last equality, we have made use of (2.1). The above equality, in view of (2.2), shows that $M > K$.

To complete the proof of the lemma, we have to show that, $M = 1$ in (2.2) if and only if $N = 1$ (and $K = 0$). Now, (2.2) with $N = 1$ (and $K = 0$) reads as

$$MSM(M + 2) - MSM(M + 1) = 3.$$

Since $MSM(3) - MSM(2) = 3$, it follows, by virtue of Corollary 2.1, that $M = 1$.

Again, if $M = 1$ in (2.2), we get

$$MSM(N + 2) - MSM(N + 1) = 3^{K+1}. \tag{2}$$

From (2.1), (2.3) and (2), we must have $K = 0$.

An implication of Lemma 2.4 is that, there are infinitely many integers N such that $MSM(N) - MSM(N-1)$ is of the form 3^K for some integer $K \geq 0$.

Given any integer N (≥ 1), by part (b) of Lemma 2.1, there is a unique integer K (≥ 1) such that $MSM(N)$ is attained at $k = K$. Now, given any integer K (≥ 1), is there an integer N (≥ 1) such that $MSM(N)$ is attained at $k = K$? The following proposition answers the question in the affirmative.

Proposition 2.1 : Given any integer $K \geq 1$, there is an integer $N \geq 1$ such that $MSM(N)$ is attained at the point $k = K \geq 1$.

Proof : The proof is by induction on K. The result is true for $K = 1$ with $N = 1$. So, we assume that the result is true for some integer $K \geq 1$, that is, we assume that, for K (≥ 1), there is an integer N such that $MSM(N)$ is attained at $k = K$, so that

$$MSM(N) = 2MSM(N - K) + \tfrac{1}{2}(3^K - 1).$$

Now, by Lemma 2.2, $MSM(N + 1)$ is attained either at $k = K$, or else, at $k = K + 1$. If $MSM(N + 1)$ is attained at $k = K + 1$, the proof by induction is complete. Otherwise, $MS(N + 1)$ is attained at $k = K$, so that

$$MSM(N + 1) = 2MSM(N - K + 1) + \frac{1}{2}(3^K - 1)$$

$$< 2MSM(N - K) + \frac{1}{2}(3^{K+1} - 1),$$

and hence

$$MSM(N + 1) - MSM(N) < 3^K.$$

Now, if $MS(N + 2)$ is attained at $k = K + 1$, the proof is complete; otherwise

$$MSM(N + 2) = 2MSM(N - K + 2) + \frac{1}{2}(3^K - 1)$$

$$< 2MSM(N - K + 1) + \frac{1}{2}(3^{K+1} - 1),$$

giving

$$MSM(N + 2) - MSM(N + 1) < 3^K.$$

Thus, $MSM(N + 1)$, $MSM(N + 2)$, …, $MSM(N + m)$, … are all attained at $k = K$, with

$$MSM(N + i) - MSM(N + i - 1) < 3^k,\ i = 1, 2, \ldots .$$

But since the sequence $\{MSM(N+i) - MSM(N+i-1\}_{n=1}^{\infty}$ is strictly increasing in i ($\geq$ 1) (by part (a) of Lemma 2.1), there is an integer m (≥ 1) such that

$$MSM(N + m) - MSM(N + m - 1) \geq 3^K.$$

For the minimum such m, say, $m = M$, $MSM(N + M - 1)$ is attained at the point $k = K$ but $MSM(N + M)$ is attained at $k = K + 1$, with

$$MSM(N + M) - MSM(N + M - 1) = 3^K.$$

Thus, corresponding to $K + 1$, there is an integer $N + M$ such that $MSM(N + M)$ is attained at $k = K + 1$, which we intended to prove.

Proposition 2.1 shows that, given any integer $K \geq 1$, there is an integer N (≥ 1) such that $MSM(N)$ is attained at $k = K$. However, note that, such N is not unique. For example, both $MSM(3)$ and $MSM(4)$ are attained at $k = 2$.

Let the sequence $\{b_n\}_{n=1}^{\infty}$ be defined by

$$a_n = MSM(n) - MSM(n-1),\ n \geq 1. \tag{2.4}$$

Let $k_j \geq 1$ be defined by

$$a_{k_j} = MSM(k_j) - MSM(k_j - 1) = 3j;\ j \geq 0 \tag{2.5}$$

with

$$k_0 = 1. \tag{2.6}$$

The Corollary below follows from Lemma 2.4, a proof of which is given in Majumdar [3].

Corollary 2.2 : For all $j \geq 0$, $MSM(k_j - 1)$ is attained at $k = j$; moreover, for all n satisfying the inequality $k_j \leq n \leq k_{j+1} - 1$, $MSM(n)$ is attained at $k = j + 1$.

Theorem 2.1 : For all $j \geq 0$ and $1 \leq s \leq k_{j+1} - k_j - 1$,

$$MSM(k_j + s) - MSM(k_j + s - 1) = 2[MSM(k_j + s - j - 1) - MSM(k_j + s - j - 2)].$$

Proof : By Corollary 2.2, $MS(k_j + s - 1)$ and $MS(k_j + s)$ both are attained at the point $k = j + 1$, so that by (1.5),

$$MSM(k_j + s - 1) = 2MSM(k_j + s - j - 2) + \tfrac{1}{2}(3^{j+1} - 1),$$

$$MSM(k_j + s) = 2MSM(k_j + s - j - 1) + \tfrac{1}{2}(3^{j+1} - 1).$$

We have the following result.

Lemma 2.5 : For all $n \geq 1$, a_n is of the form $2^i 3^m$ for some integers $i \geq 0, m \geq 0$.

Proof : Cleary, the result is true when $n = 1$ (with $i = 0$, $m = 0$). To proceed by induction, we assume that the result is true for all $t \leq n$.

To prove the result for $n + 1$, we have to consider, by Lemma 2.2, the following two possibilities :

Case 1 : $MSM(n)$ and $MSM(n + 1)$ both are attained at $k = K$. In this case,

$$a_{n+1} = 2^{N-K};$$

Case 2 : $MSM(n)$ is attained at $k = K$ and $MSM(n + 1)$ is attained at $k = K + 1$. Here,

$$a_{n+1} = 2[MSM(K + 1) - MSM(K)] = 2\,a_{K+1}.$$

Now, by the induction hypothesis, a_{K+1} is of the form $2^i 3^m$, and hence, so is a_{n+1}.

Lemma 2.5 shows that, for any $n \geq 1$, a_n (defined through the equation (2.4)) is of the form $2^i 3^m$ for some integers $i \geq 0$ and $m \geq 0$. In the next section, we prove the converse, namely that, given any integer of the form $2^i\, 3^m$, there is an integer n such that $a_n = 2^i\, 3^m$.

3. Main Results

Let $\{b_n\}_{n=1}^{\infty}$ be the sequence of integers, arranged in (strictly) increasing order :

$$b_n = 2^i\ \ 3^m,\ i \geq 0,\ m \geq 0. \tag{3.1}$$

The first few terms of $\{b_n\}_{n=1}^{\infty}$ are

$$1, 2, 3, 4, 6, 8, 9, 12, 16, 18, 24, \ldots,$$

and are known as *3-smooth numbers*.

Given any integer $j \geq 0$, we first derive the number of elements of the sequence $\{b_n\}_{n=1}^{\infty}$ such that $3^j < b_n < 3^{j+1}$, that is

$$3^j < 2^i\, 3^m < 3^{j+1}. \tag{3}$$

Clearly, i must satisfy the inequality :

$$(j - m)\, \frac{ln3}{ln2} < i < (j - m + 1)\, \frac{ln3}{ln2}. \tag{4}$$

From (3), we observe that, when $j = 0$, $m = 0$.

Let $N(n, j)$ be the number of elements of the sequence $\{b_n\}_{n=1}^{\infty}$ satisfying the inequality (3), that is,

$$N(n,\ j) = \left| \left\{ b_n : 3^j < b_n < 3^{j+1} \right\} \right| = \left| \left\{ b_n : 3^j < 2^i 3^m < 3^{j+1} \right\} \right|. \tag{5}$$

Then, we have the following lemma, due to Majumdar [3], which gives the recurrence relation satisfied by $N(n, j)$.

Lemma 3.1 : For any integer $j \geq 1$,

$$N(n,\ j+1) = N(n,\ j) + \left| \left\{ i\ :\ 3^{j+1} < 2^i < 3^{j+2} \right\} \right|.$$

Corollary 3.1 : For any integer $j \geq 0$,

$$N(n, j) = \left| \left\{ b_n : 3^j < b_n < 3^{j+1} \right\} \right| = \max\left\{ i : 2^i < 3^{j+1} \right\} = \left\lfloor (j+1) \frac{\ln 3}{\ln 2} \right\rfloor$$

where $\lfloor x \rfloor$ is the floor of the real number $x > 0$.

Proof : The left-side part of the above chain of equalities has been proved in Majumdar [3]. Now, since

$$2^i < 3^{j+1} \text{ if and only if } i\, ln2 < (j + 1)\, ln\, 3,$$

the remaining part follows.

Next, we find the number of elements of the sequence $\{b_n\}_{n=1}^{\infty}$ such that $3^j < b_n < 2.3^j$ for any integer $j \geq 0$, that is,

$$3^j < 2^i 3^m < 2.3^j. \tag{6}$$

Lemma 3.2 : For any integer $j \geq 0$,

$$\left| \{b_n : 3^j < b_n < 2.3^j\} \right| = \left| \{b_n : 3^j < 2^i 3^m < 2.3^j\} \right| = j.$$

Proof : First note that the inequality (6) is satisfied if and only if

$$(j-m)\frac{ln3}{ln2} < i < (j-m)\frac{ln3}{ln2} + 1. \tag{7}$$

When $j = 0$, $m = 0$, and the result is true. So, let $j \geq 1$. In this case, the inequality (7) admits j number of solutions, corresponding to $m = 0, 1, \ldots, j-1$.

Since

$$|(j + 1)\frac{ln3}{ln2}| > j \text{ for all } j \geq 1,$$

from Corollary 3.1 and Lemma 3.2, we see that

$$\left| \{ b_n : 3^j < b_n < 2.3^j \} \right| < \left| \{ b_n : 3^j < b_n < 3^{j+1} \} \right|, \text{ if } j \geq 1.$$

Lemma 3.3 : For any integer $j \geq 0$,

$$\left| \{ a_n : 3^j < a_n < 2.3^j \} \right| = \left| \{ b_n : 3^j < b_n < 2.3^j \} \right| = j.$$

Proof : In Lemma 2.4, let

$$N = k_j, K = j.$$

Then, (2.3) reads as

$$a_{k_j+j+1} = 2.3^j.$$

Now, the number of elements of the sequence $\{b_n\}_{n=1}^{\infty}$ between a_{k_j} and a_{k_j+j+1} is j. This, coupled with Lemma 3.2, gives the result desired.

Let the sequence of numbers $\{p_j\}_{j \geq 0}$ be defined as follows :

$$b_{p_j} \equiv 3^j, j \geq 0. \tag{3.2}$$

Clearly,

$$p_0 = 1. \tag{3.3}$$

The following lemma gives a recurrence relation satisfied by $\{b_n\}_{n=1}^{\infty}$

Lemma 3.4 : For any n such that $3^j < b_n < 3^{j+1}$ for some integer $j \geq 0$, $b_n = 2b_{n-j-1}$.

Proof : When $j = 0$, $1 < b_1 < 3$, and

$$b_2 = 2b_1 = 2\, b_{2-o-1},$$

and the result is true. So, we assume that the result is true for (and up to) some j, that is, we assume that, for all n such that $3^{s-1} < b_n < 3^s$, $1 \leq s \leq j$,

$$b_n = 2\, b_{n-s}.$$

To prove the result for $j + 1$, we assume that

$$b_p \equiv 3^j < b_n < 3^{j+1} \equiv b_p$$

Clearly, any such b_n is even. Also, since $3^j < 2.3^j < 3^{j+1}$, it follows that, for any N with $3^j < b_N < 2.3^j$, $b_N = 2\, b_M$, for some integer M with $3^{j-1} < b_M < 3^j$. Then, by the induction hypothesis, $b_M = 2\, b_{M-j}$.

Now,

$$\left| \left\{ b_n : \frac{b_M}{2} < b_n < \frac{b_N}{2} \right\} \right| = \left| \left\{ b_n : b_M < b_n < b_N \right\} \right| = N - M - 1.$$

Therefore,

$$\frac{b_N}{2} = b_{M\text{-}j+(N\text{-}M\text{-}1)} = b_{N\text{-}j\text{-}1},$$

so that

$$b_N = 2\, b_{N-j-1};\ 3^j < b_N < 2.3^j.$$

Let L be the maximum such N so that

$$b_L = 2\, b_{L-j-1},\ b_{L+1} = 2.3^j.$$

Now, since (by Lemma 3.2), there are j elements of the sequence $\{b_n\}_{n=1}^{\infty}$ such that

$$b_{p_j} < b_n < 2b_{p_j},$$

it follows that

$$L - p_j = j,$$

so that

$$b_{L+1} = 2\, b_{L-j}.$$

Thus, the result is true for all n with $3^j < b_n \leq 2.3^j$.

If $j \geq 1$, there is at least one b_n such that $2b_{p_j} < b_n < b_{p_{j+1}}$, the minimum of which is b_{L+2}.

Clearly,

$$b_{L+2} = 2b_{p_{j+1}} \equiv 2b_{L-j+1}.$$

In general, if $b_{L+s} < b_{p_{j+1}}$, $2 \le s \le \left\lfloor (j+1)\frac{ln3}{ln2} \right\rfloor - j - 1$, then $b_{L+s} = 2\, b_{L+s-j-1}$.

We now prove the main result of the paper, given in the theorem below.

Theorem 3.1 : For all $n \geq 1$, $a_n = b_n$.

Proof : Let $k_j \leq n < k_{j+1}$ for some integer $j \geq 0$ (so that $3^j \leq a_n < 3^{j+1}$, $3^j \leq b_n < 3^{j+1}$).

Then, by Theorem 2.1,

$$a_{k_j+s} = 2a_{k_j+s-j-1} \text{ for all } 1 \leq s \leq k_{j+1} - k_j - 1.$$

Thus, by virtue of Lemma 3.3, a_n and b_n satisfy the same recurrence relation, and hence, we get the desired result.

The following theorem gives the solution of the recurrence relation (1.2).

Theorem 3.2 : For $n \geq 1$,

$$MSM(n) = \sum_{m=1}^{n} a_m = \sum_{m=1}^{n} b_m.$$

Moreover, if $3^j \leq a_n < 3^{j+1}$ for some integer $j \geq 0$ (so that $k_j \leq n < k_{j+1}$), $MSM(n)$ is attained at the unique point $k = j + 1 = \left\lfloor \frac{ln(b_n)}{ln3} \right\rfloor + 1$.

Proof : Since $MSM(n)$ can be written as

$$MSM(n) = \sum_{m=1}^{n} \left[MSM(m)\text{-}MSM(m\text{-}1) \right],$$

we get the desired expression of $MSM(n)$ by Theorem 3.1. Also, by Corollary 2.2, $MSM(n)$ is attained at the unique point $k = j + 1$. Now, since $3^j \leq b_n < 3^{j+1}$, we see that j must satisfy the inequality $j\, ln3 \leq ln(b_n) < (j + 1)\, ln3$.

We then get the desired expression of k, given in the lemma.

A consequence of Theorem 3.2 is the following

Corollary 3.2 : For all $j \geq 0$,

$$k_{j+1} = k_j + \left\lfloor (j+1)\frac{ln3}{ln2} \right\rfloor + 1,\ k_0 = 1.$$

Proof : follows immediately, since (by Corollary 3.1), the number of elements of the sequence $\{ b_n \}_{n=1}^{\infty}$ on (k_j, k_{j+1}) is $\left\lfloor (j+1)\frac{ln3}{ln2} \right\rfloor$.

Since $k_0 = 1$, Corollary 3.2 allows us to find k_j recursively in j. Table 3.1 below gives the values of k_j for some small values of k, calculated on a computer, using the recurrence relation in Corollary 3.2.

Table 1 : Values of k_j for $0 \le j \le 8$

j	0	1	2	3	4	5	6	7	8
k_j	1	3	7	12	19	27	37	49	62

From Table 3.1, we see that $b_3 = k_1 = 3^1 = 3$, $b_7 = k_2 = 3^2 = 9$. Moreover, there is only one element of the sequence $\{b_n\}_{n=1}^{\infty}$ between k_1 and k_0.

Corollary 3.2 is a new result which, together with Lemma 3.4, enables us to find b_n recursively for any fixed n. For example, to find b_{17}, we proceed as follows : Looking at Table 3.1, since $12 < 17 < 19$, it follows that $j = 3$ in Corollary 3.2, and so, by Lemma 3.4, $b_{17} = 2\, b_{13}$.

Thus, we need to find b_{13}. From Table 3.1, we see that $j = 3$ in Corollary 3.2, and so by Lemma 3.4, $b_{13} = 2\, b_9$.

To find b_9, from Table 3.1, we see that $j = 2$ in Corollary 3.2, so that $b_9 = 2\, b_6$.

Similarly, $b_6 = 2\, b_4 = 8$.

Plugging in the values of b_7 and b_{10}, we finally get, $b_{17} = 64$.

Corollary 3.3 : For all $n \ge 1$, $MS(n) = \sum_{m=1}^{n} a_m = 2 \sum_{m=1}^{n} b_m$.

Moreover, letting $c_n = MS(n) - MS(n-1),\ n \ge 1$,

if $2.3^j \le c_n < 2.3^{j+1}$ for some integer $j \ge 0$, $MS(n)$ is attained at the unique point $k = j + 1 = \left\lfloor \frac{ln(b_n)}{ln 3} \right\rfloor + 1.$

Proof : follows by virtue of Theorem 3.2, together with (1.3) and Lemma 1.1.

The expression of $MS(n)$ is given in Corollary 3.3 above. As has been pointed out by Stockmeyer [6] (without proof), an interpretation of the expression of $MS(n)$ is as follows : In the star puzzle with n discs $d_1, d_2, \ldots, d_n$ in increasing order (so that d_1 is the smallest disc and d_n is the largest one), the disc d_{n-m+1} needs exactly $2\, b_m$ number of moves ($1 \le m \le n$) under the optimal strategy.

4. Concluding Remarks

Stockmeyer [6] gives an outline of a proof of the results in Corollary 3.3, but his argument is heuristic, is not supported by any theoretical development, and is incomplete in the sense that it lacks the proof that

$$MS(n) - MS(n-1) = 2b_n.$$

This paper gives a rigorous treatment of the problem, which unveils many interesting properties. Lemma 3.4 gives a new recurrence relation satisfied by the sequence $\{b_n\}_{n=1}^{\infty}$ and Corollary 3.2 gives a recurrence relation satisfied by k_j. It is interesting that $MSM(n)$ satisfies the same recurrence relation as that of b_n.

From Corollary 3.3, we observe that, it in fact offers two methods of finding $MS(n)$ (for any $n \geq 1$). The first method is in terms of the sequence $\{b_n\}_{n=1}^{\infty}$. When n is small, we can readily find $MS(n)$ by this method. For example, adding the first 6 terms of the sequence $\{b_n\}_{n=1}^{\infty}$, we see that $MS(6) = 48$. For large n, we may use Lemma 3.4 to find b_n. Thus, for example, to find $MS(17)$, by Corollary 3.3, we have

$$MS(17) = 2[b_1 + b_2 + b_3 + (b_4 + b_5 + b_6) + b_7 + (b_8 + b_9 + b_{10} + b_{11})$$
$$+ b_{12} + (b_{13} + b_{14} + b_{15} + b_{16} + b_{17})]$$

Now,

$$b_8 = 2b_5,\ b_9 = 2b_6,\ b_{10} = 2b_7,\ b_{11} = 2b_8,$$
$$b_{13} = 2b_9,\ b_{14} = 2b_{10},\ b_{15} = 2b_{11},\ b_{16} = 2b_{12},\ b_{17} = 2b_{13},$$
$$b_7 = 3^2 = 9,\ b_{12} = 3^3 = 27.$$

Thus,

$$MS(17) = 2[1 + 2 + 3 + (4 + 6 + 8) + 9 + (12 + 16 + 18 + 24)$$
$$+ 27 + (32 + 36 + 48 + 54 + 64)] = 728.$$

Alternatively, from Corollary 3.3, since $MS(17)$ is attained at $k = 3 + 1 = 4$, we get

$$MS(17) = 2MS(13) + 3^4 - 1 = 2MS(13) + 80.$$

Now, by Corollary 3.3 again, $MS(13)$ is attained at $k = 3 + 1 = 4$, so that

$$MS(13) = 2MS(9) + 3^4 - 1 = 2MS(9) + 80.$$

Since $MS(9)$ is attained at $k = 3$, we get

$$MS(9) = 2MS(6) + 3^3 - 1 = 2MS(6) + 26.$$

Now, $MS(6)$ is attained at $k = 2$, and so

$$MS(6) = 2MS(4) + 3^2 - 1 = 40 + 8 = 48.$$

Finally, we get $MS(17) = 728$.

The following recurrence relation has been considered by Matsuura [4] :

$$T(n, \alpha) = \min_{0 \leq k \leq n-1} \left\{ T(k, \alpha) + 2^{n-k} - 1 \right\},\ n \geq 1,$$

$$T(0, \alpha) = 0 \text{ for all } \alpha \geq 3,$$

where $\alpha \geq 3$ is an integer. The problem was taken up by Majumdar [2], who studied some of the properties of $T(n, \alpha)$. Of particular interest is $T(n, 3)$. It has been shown by Matsuura [4], by induction on n, that

$$T(n, 3) - T(n-1, 3) = b_n \text{ for all } n \geq 1,$$

so that $MSM(n) = T(n, 3)$ for all $n \geq 1$.

In passing, it may be mentioned here that, a second recurrence relation satisfied by the sequence $\{b_n\}_{n=1}^{\infty}$ has been derived by Matsuura [4], and is given below.

Lemma 4.1 : For any n such that $2^i < b_n < 2^{i+1}$ for some integer

$$i \geq 0.\ b_n = 3b_{n-i-1}.$$

To make the above result applicable, it is to be supplemented by Lemma 4.3. To prove Lemma 4.3, we need the result below.

Lemma 4.2 : For any integer $i \geq 0$,

$$\left| \left\{ b_n : \ 2^i < b_n < 2^{i+1} \right\} \right| = \left\lfloor (i+1)\frac{ln2}{ln3} \right\rfloor .$$

Proof : is similar to that of Corollary 3.1, and is left as an exercise.

Lemma 4.3 : Let the sequence of integers $\{q_i\}_{i \geq 0}$ be defined as follows :

$$b_{q_i} = 2^i,\ i \geq 0;\ q_0 = 1.$$

Then,

$$q_{i+1} = q_i + \left\lfloor (i+1)\frac{ln2}{ln3} \right\rfloor + 1.$$

Proof : is similar to Corollary 3.2, and is omitted here.

Using Lemma 4.3, we can form the table below, which would help us in finding b_n for any $n \geq 1$ fixed.

Table 4.1 : Values of q_i for $0 \leq i \leq 8$

i	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
q_i	1	2	4	6	9	13	17	22	28

Thus, for example, $q_1 = 2$, so that $b_2 = 2$; similarly, since $q_2 = 4 = 2^2$, it follows that $b_4 = 4$.

Using Lemma 3.4, we found that b_{17} = 72. If we apply Lemma 4.3, we see from the above Table 4.1 that, $b_{17} = 2^6$ = 64.

From the computational point of view, to find b_n for $n \geq 1$ fixed, the recurrence relation in Lemma 4.3 is more efficient than that in Lemma 3.4.

REFERENCES

[1] Majumdar, A.A.K. (2013). The Classical Tower of Hanoi Problem and Its Generalizations, Vol. 2 : Other Generalizations, Lambert Academic Publishing, USA.

[2] Majumdar, A.A.K. (2016),Some Local-Value Relationships for the Recurrence Relation Related to the Tower of Hanoi Problem. *Proceedings of the Pakistan Academy of Sciences, A. Physical and Computational Sciences* 53(2)**:** 187-201.

[3] Majumdar, A.A.K. (2017). Some Local-Value Relationships Related to the Star Puzzle. *Proceedings of the Pakistan Academy of Sciences, A. Physical and Computational Sciences* 54(2)**:** 219-228.

[4] Matsuura, A. (2008). Exact Analysis of the Recurrence Relations Generated from the Towers of Hanoi, *SIAM Proceedings in Applied Mathematics* 129**:** 228-233.

[5] Scorer, J.S., Grundy, P.M. and Smith, C.A.B. (1944). Some binary games. *The Mathematical Gazette* 280**:** 96-103.

[6] Stockmeyer, P.K. (1994). Variations on the four-post Tower of Hanoi puzzle. *Congressus Numerantium* 102**:** 3-12.