EFFECT OF RADIATION AND HEAT GENERATION ON 
UNSTEADY MHD FLOW OF NON-NEWTONIAN FLUID 
ALONG A STRETCHING VERTICAL SURFACE

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ABSTRACT

In this paper, the unsteady forced convection boundary-layer flow of a non-Newtonian fluid along a continuously moving stretching sheet with thermal radiation and heat generation or absorption in the presence of magnetic field has been studied. First, the governing equations have been nondimensionalized by usual transformations to obtain the similar solutions. Then, the obtained equations have been solved by an implicit finite difference method. The convergency of explicit and implicit finite difference method has also been discussed. The results are presented for the effect of various parameters such as magnetic parameter ($M$), radiation parameter ($N$), heat source parameter ($Q$), Prandtl number ($Pr$) and power-law fluid index ($n$). The effects of these parameters on skin-friction coefficient ($C_f$) and the local Nusselt number ($Nu$) which are of physical and engineering interest have also been studied and presented graphically.

Keywords: Unsteady MHD flow, Non-Newtonian fluid, Heat generation/absorption, Radiation

1. Introduction

Over recent years, applications of non-Newtonian fluids in many industrial processes have been an interesting topic to many researchers. Different models have been proposed to explain the behavior of non-Newtonian fluid. Schowalter [1] was the first one, who formulated the boundary layer flow of a Non-Newtonian fluid. The study of flow and heat transfer generated by stretching surface plays a significant role in many material processing applications such as hot rolling, extrusion, metal forming, wire and glass fiber drawing and continuous casting. Numerous investigations like Banks [2] and Chen [3] were done on the stretching sheet problem with linear stretching in different directions in the absence of the magnetic field.

The study of effects of magnetic field on free convection flow is important in liquid-metals, electrolytes and ionized gases. Kishan and Sashidhar [4] analyzed the momentum and heat transfer in laminar boundary layer flow of non-Newtonian fluids past a semi-infinite flat plate with the thermal dispersion in the presence of a uniform magnetic field for both the cases of static plate and continuous moving plate. However, at present, the radiation effect on MHD flow and heat transfer problems has become more important industrially.
Effects of radiation on non-Newtonian fluids have been studied by many authors like Sahoo and Poncet [5] and Sajid and Hayat [6]. All the above mention studies consider the steady-state problem. But in certain practical problems, the motion of the stretched surface may start from the rest. In these problems, the transient or unsteady aspects become more interesting. The unsteady heat transfer problems over a stretching surface, which is started impulsively from restore is stretched with a velocity that depends on time, are considered. Elbashbeshy and Bazid [7] presented an exact similarity solution for unsteady momentum and heat transfer flow whose motion is caused solely by the linear stretching of a horizontal stretching surface. In the case of unsteady boundary-layer flow, Singh [8] investigated the thermal radiation and magnetic field effects on an unsteady stretching permeable sheet in the presence of free stream velocity. Later, Gamal and Abdel-Rahman [9] analyzed the effects of variable viscosity and thermal conductivity on unsteady MHD non-Newtonian flow over a stretching porous sheet.

In this present study, we have studied the effects of forced convection on unsteady magneto-hydrodynamic (MHD) non-Newtonian fluid flow along a vertical, continuously moving stretching sheet with radiation and heat generation or absorption. Firstly, the governing nonlinear partial differential equations are transformed into a system of dimensionless nonlinear partial differential equations using appropriate transformations. The resulting non-linear equations are solved numerically using both an explicit and an implicit finite difference technique. A comparison is also presented between these two schemes based on the convergence. Later, a parametric study has been carried out to explore the effect of some physically important parameters such as magnetic parameter ($M$), radiation parameter ($N$), heat source parameter ($Q$), Prandtl number ($Pr$) and power-law fluid index ($n$) on momentum and thermal boundary layers. The effects of these parameters on skin-friction coefficient ($C_f$) and the local Nusselt number ($Nu$) have also been discussed.

2. Mathematical Analysis

Consider the unsteady MHD forced convection laminar boundary layer flow of a viscous, incompressible non-Newtonian fluid past a continuously moving stretched surface in the vertical direction under the influence of thermal radiation and heat generation or absorption.

The frame of reference is chosen such that stretching surface coincides with the plane $Y$ where the flow is assumed only for $Y > 0$. Introducing the Cartesian co-ordinate system, the X-axis is taken along the stretching surface in the vertically upward direction and the Y-axis is taken as normal to the surface. Two equal and opposite forces are introduced along the X-axis so that the surface is stretched, keeping the origin fixed.

An external magnetic field is applied in the Y-direction. Since we have taken the fluid to be the electrically conducting, hence we can neglect the effect of the induced magnetic field in comparison to the applied magnetic field. Therefore, only the applied magnetic field $B_0$ plays a
role which gives rise to magnetic force $F_x = \frac{\sigma B_0^2 u}{\rho}$ in the X-direction.

Since, we are considering forced convection, there is no influence of temperature field on velocity field i.e. there is no bouncy force. The Rosseland approximation (Rohsenow [10]) is used to describe the radioactive heat flux $q_x$ in the energy equation. The radioactive heat flux in the X-direction is considered negligible in comparison to the Y-direction. The physical configuration and coordinate system is shown in Figure 1.

Under the above assumptions and the usual boundary layer approximation, the unsteady MHD forced convection non-Newtonian fluid flow with heat and the radiation effect are governed by the following equations:

**Continuity equation:**

$$\frac{\partial v_y}{\partial y} = 0$$

(1)

**Momentum equation:**

$$\rho \left( \frac{\partial u}{\partial t} + v_0 \frac{\partial u}{\partial y} \right) = K \frac{\partial}{\partial y} \left( \frac{\partial u}{\partial y} \right) - \sigma B_0^2 u$$

(2)
Energy equation:

\[
\rho C_v \left( \frac{\partial T}{\partial t} + \nu_0 \frac{\partial T}{\partial y} \right) = \kappa \frac{\partial^2 T}{\partial y^2} + Q_o (T - T_o) - \frac{\partial q_r}{\partial y} \quad (3)
\]

The initial and boundary conditions are:

**Initial conditions:**
\[
u = 0, \quad T = 0 \quad \text{at} \quad t = 0
\quad (4)

**Boundary conditions:**
\[
u = U_u, \quad T = T_u \quad \text{when} \quad y = 0 \quad \text{at} \quad t > 0
\]
\[
u = 0, \quad T = T_u \quad \text{when} \quad y \to \infty \quad (5)
\]

Where, \(\nu\) and \(\nu_0\) are the velocity components in the X and Y direction respectively, \(t\) is for time, \(T\) represents temperature field, \(U_o\) and \(T_o\) are velocity and temperature of free stream, \(T_u\) is temperature at the surface, \(\rho\) is the fluid density, \(\kappa\) is the consistency coefficient, \(\sigma\) is the electric conductivity, \(B_o\) is the magnetic field, \(n\) is the power-law fluid index, \(C_p\) is the specific heat at constant pressure, \(\kappa\) is the thermal conductivity of the fluid, \(Q_o\) is the volumetric rate of heat generation or absorption and \(q_r\) is the radioactive heat flux.

By using the Rosseland approximation, the radioactive heat flux \(q_r\) is,

\[
q_r = \frac{4\sigma_e \nu T^4}{k_e} \quad (6)
\]

Where, \(\sigma_e\) is the Stefan–Boltzmann constant and \(k_e\) is the mean absorption coefficient.

If the temperature differences within the flow are sufficiently small, then Eq.(6) can be linearized by expanding \(T^4\) in a Taylor series about \(T_u\), which after neglecting higher order terms takes the form,

\[
T^4 \approx 4 T_u^3 T - 3 T_u^4 \quad (7)
\]

Therefore, using (6) and (7), we have

\[
\frac{\partial q_r}{\partial y} = - \frac{16\sigma_e T_u^3}{3k_e} \frac{\partial^2 T}{\partial y^2} \quad (8)
\]

Finally, using (8) in the energy equation (3), the governing boundary layer equations become,

**Continuity equation:**
\[
\frac{\partial \nu}{\partial y} = 0 \quad (9)
\]

**Momentum equation:**
\[
\frac{\partial \nu}{\partial t} + \frac{\partial \nu}{\partial y} = \frac{\kappa}{\rho} \frac{\partial}{\partial y} \left( \frac{\partial \nu}{\partial y} \right) - \frac{\sigma}{\rho} \frac{B_o^2 U}{\rho} \quad (10)
\]

**Energy equation:**
\[
\frac{\partial T}{\partial t} + \nu_0 \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} + \frac{Q_o}{\rho C_p} (T - T_u) + \frac{16\sigma_e T_u^3}{k_e \rho C_p} \frac{\partial^2 T}{\partial y^2} \quad (11)
\]

Subject to the initial condition (4) and boundary condition (5). Where, \(\alpha\) is the thermal diffusivity. To make the governing equations dimensionless introducing the following dimensionless variables,

\[
Y = \frac{y}{L}, \quad U = \frac{\nu}{U_o}, \quad V = \frac{\nu_0}{U_o}, \quad \bar{t} = \frac{t U_o}{L}, \quad \bar{T} = \frac{T - T_u}{T_u - T_o} \quad (12)
\]
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Where, L is the characteristic length.

The nonlinear coupled partial differential equations in terms of dimensionless variables are,

**Continuity equation:**
\[
\frac{\partial V}{\partial Y} = 0
\]

**Momentum equation:**
\[
\frac{\partial U}{\partial t} + V \frac{\partial U}{\partial Y} = \frac{1}{\text{Re}} \frac{\partial}{\partial Y} \left( \frac{\partial U}{\partial Y} \right)^{-1} \frac{\partial U}{\partial Y} - MU
\]

**Energy equation:**
\[
\frac{\partial T}{\partial t} + V \frac{\partial T}{\partial Y} = \frac{1}{\text{Re Pr}} \left( 1 + \frac{4}{3N} \right) \frac{\partial^2 T}{\partial Y^2} + Q\overline{T}
\]

With initial and boundary equations:

**Initial conditions:**
\[ U = 0, \quad T = 0 \] at \( \tilde{t} = 0 \)

**Boundary conditions:**
\[ U = 1, \quad T = 1 \quad \text{when} \quad Y = 0 \]
and \[ U = 0, \quad T = 0 \quad \text{when} \quad Y \to \infty \] at \( \tilde{t} > 0 \)

Where, the local Reynolds number, \( \text{Re} = \frac{U_0^2 \alpha \rho L}{K} \), the heat source parameter, \( Q = \frac{Q_n L}{\rho C_p U_0} \), the generalized Prandtl number, \( \text{Pr} = \frac{K/ \rho}{\alpha \left( \frac{U_0}{L} \right)^{n-1}} \), the radiation number, \( N = \frac{k \alpha}{4\sigma T_o^3} \), and the magnetic field parameter, \( M = \frac{\sigma b_s^2 L}{\rho U_0} \).

Dimensionless skin friction coefficient \( (C_f) \) and the local Nusselt number \( (N_u) \) are,
\[
C_f = \frac{\tau}{\frac{1}{2} \rho U_0^2} \quad \text{or} \quad C_f = \frac{2}{\text{Re}} \left. \frac{\partial U}{\partial Y} \right|_{\tilde{t}=0}
\]
\[
N_u = \frac{hL}{\kappa} \quad \text{or} \quad N_u = -\left. \frac{\partial \overline{T}}{\partial Y} \right|_{t=0}
\]

3. **Numerical Computation**

For solving the governing equations (13)-(15) subject to the initial conditions (16) and boundary conditions (17) finite difference method is used. Both explicit and implicit difference scheme are used to approximate the solution of momentum equation (14) and only explicit difference scheme is used to approximate the solution of energy equation (15).

For this, a rectangular region of the flow field is chosen, where the X-axis is taken along the stretching surface and the Y-axis is normal to the plate. Grids along Y-axis represent time steps. Here, we consider that the length of the sheet is \( X_{\text{max}} = 20 \), i.e. \( x \text{ varies from 0 to } X_{\text{max}} \). There are \( N_x = 200 \) and \( N_y = 1000000 \) grid spacing in the X and Y directions, respectively. The mesh size along X and Y-axis are \( \Delta x \) and \( \Delta t = k = 0.000001 \) respectively. From continuity equation (13) we can consider \( V \) as a constant velocity.

For solving momentum equation in explicit difference technique, we will use a 2-point forward
difference approximation for time derivative and central difference approximation for spatial derivatives. While in implicit finite difference method, 2-point backward difference approximation is used to approximate time derivative and central difference approximations are used to approximate spatial derivatives. Here, \( \frac{\partial U}{\partial Y} \) is approximated by the 2-point central difference approximation at the previous time step to avoid nonlinearity. The obtained system of linear equations is solved using Gauss-Seidel iterative technique.

For approximating the solution of the energy equation, 2-point forward difference approximation is used for time derivative and central difference approximation for spatial derivatives.

![Fig. 2: Velocity error convergency profile in explicit and implicit method.](image)

The same values of all the parameters and constants have been used in both techniques. Here we have acquired a convergency of \( 10^{-4} \). In implicit technique, the error converges exponentially to zero almost after time steps \( j = 1000 \) while in explicit method after numerous fluctuation the error converges to zero after time steps \( j = 22000 \) (Fig. 2). Therefore, velocity error converges to zero faster in implicit method than in explicit method. Now we will do our further analysis by using implicit technique.

4. Results and Discussion

The system of non-linear partial differential equations (13)-(15) are solved under initial and boundary conditions (16) and (17) by using implicit finite difference scheme along with Gauss-Seidel iteration method. Numerical results are obtained for various values of magnetic parameter (M), radiation parameter (N), Prandtl number (Pr), heat source parameter (Q) and power-law fluid index (\( n \)) to study the effects of these parameters on momentum and thermal boundary layers as well as on skin friction coefficient \( (C_f) \) and the local Nusselt number \( (Nu_L) \). The graphs are plotted for dimensionless velocity \( (U) \) and temperature \( (T) \) at dimensionless time \( \tilde{t} = 1 \).

Fig. 3 represents the velocity distribution for different values of Prandtl number \( (Pr = 0.71, \ 1, \ 2, \ 7, \ 9, \ 11) \), radiation parameter \( (R = 0.1, \ 0.5, \ 1, \ 3, \ 5, \ 7) \) and heat source parameter.
Effect of Radiation and Heat Generation on Unsteady MHD Flow of Non-Newtonian Fluids: 

For dilatants, Newtonian fluids and pseudo plastics, it is observed that velocity profile doesn’t change with the change of parameters $Pr$, $Q$ and $N$ but increases as the power-law fluid index $n$ increases.

- Velocity profile decreases as magnetic parameter $M$ increases for all values of power-law fluid index $n$.

$$n = 0.5, 1, 1.5$$

- Velocity profiles decreases as magnetic parameter $M$ increases for all values of power-law fluid index $n$ (Fig. 4). It is seen that the presence of magnetic field causes higher restriction to the fluid, which has reduced the fluid velocity. There is a steep fall in velocity in pseudo plastics $n < 1$ compare to other two types.
Temperature profile decreases as radiation parameter \(N\) increases (Fig. 5). We observe that, temperature profile reduces rapidly almost at an average rate of 5% with every increase of \(N\) from 0.1 up to 1. But as we increase \(N\) further, decreasing rate slows down. Due to radiation temperature rises quickly near the wall but then decreases immediately. Thus, temperature boundary layer can be controlled effectively using radiation parameter \(N\). Furthermore, heat transfer along the stretching sheet is faster when radiation is small.

**Fig. 4:** Effect of magnetic parameter (M) on velocity profile for different values on power law fluid index (n)

**Fig. 5:** Temperature profile for different values of radiation parameter (N) and power law fluid index (n)

**Fig. 6:** Temperature profile for different values of Prandtl number (Pr) and power law fluid index (n)
Effect of Radiation and Heat Generation on Unsteady MHD Flow of Non-Newtonian Fluids

In forced convection, temperature profiles decrease with the increase of Prandtl number ($Pr$) (Fig. 6). If we increase $Pr$ from 0.71, the stretching sheet gain temperature from the environment but the heat gaining rate decreases. Every time we increase $Pr$, the temperature profile drops at an average rate 1.5%. When $Pr \geq 7$, heat gaining rate is almost zero that is thermal boundary layer doesn’t change appreciably. However, temperature rises at the beginning of the flow for $Pr \geq 7$. Temperature profiles increase very rapidly, overshoot and stabilize as heat source parameter ($Q$) increases (Fig. 7). The dotted line represents the case without any heat generation or absorption. For $Q \geq 3$, temperature rises 2.5 times higher with every increase of $Q$. Due to heat generation initial wall temperature is very high. So that, the heat transfer rate is very low. In forced convection, temperature field remains unaffected by the change of magnetic parameter ($M$) (Fig. 8) and power-law fluid index ($n$) (Fig. 5-8).

Fig. 7: Temperature profile for different values of heat source parameter ($Q$) and power law fluid index ($n$)

Fig. 8: Temperature profile for different values of magnetic parameter ($M$) and power law fluid index ($n$)
Skin friction coefficient $C_f$ decreases as both $M$ and $n$ increase (Fig. 9(a)). There is a cross flow near $M = 0.5$. Pseudo plastics have the least friction when magnetic parameter is high. Since in forced convection, the parameters $Pr$, $Q$, and $N$ showed no variation on velocity field (Fig. 3), we get a constant skin friction which is $C_f = -0.19993$ for all values of $Pr$, $Q$, and $N$.

The local Nusselt number $Nu_L$ decreases as $Q$ (Fig. 9(b)), $P$ (Fig. 9(c)) and $Pr$ (Fig. 9(d)) increase and remains constant to the change of $n$ and $N$. Since there was no variation in temperature field (Fig. 8) due to $M$, $Nu_L$ remains unchanged for the variation of $M$. The value of $Nu_L$ is $-0.57486$. Negative value of $Nu_L$ represents the heat absorption at the wall.

5. Conclusion

From the present study, the concluding remarks have been taken as follows:

- Prandtl number ($Pr$), heat source parameter ($Q$) and radiation parameter ($N$) have negligible effect on velocity profile because in forced convection, velocity field is independent of temperature field but it increases with the increase of power law fluid index ($n$). Velocity profile decreases as magnetic parameter ($M$) increases for all values of $n$.

- Temperature profile increases as $Q$ increases, decreases with the increase of both $Pr$ and $N$ and remains constant to the change of $M$ and $n$.

- Skin friction coefficient ($C_f$) decreases as both $M$ and $n$ increase. ($C_f$) is same for all values of parameters $Pr$, $Q$, and $N$.

- The local Nusselt number ($Nu_L$) decreases as the parameters $Pr$, $Q$, $Q$ and $N$ increase. ($Nu_L$) is same for all values of $M$ and $n$.

REFERENCES
Effect of Radiation and Heat Generation on Unsteady MHD Flow of Non-Newtonian Fluids

# NOMENCLATURE

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<td>$K$</td>
<td>The consistency coefficient, $q_r$</td>
<td>$q_r$</td>
<td>The radioactive heat flux</td>
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<td>Free stream velocity $C_f$</td>
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**Greek symbols**

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<td>$\rho$</td>
<td>The density of the fluid $\alpha$</td>
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<td>$\sigma$</td>
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<td>$\kappa$</td>
<td>The thermal conductivity of the fluid $\sigma$</td>
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**Subscripts**

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