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CONVERGENCE OF THE NEWTON-TYPE METHOD FOR GENERALIZED EQUATIONS

M. H. Rashid[1] and A. Sarder
Department of Mathematics, Faculty of Science, University of Rajshahi
Rajshahi-6205, Bangladesh
[1]Corresponding author: harun_math@ru.ac.bd,

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ABSTRACT

Let X and Y be real or complex Banach spaces. Suppose that $f: X \to Y$ is a Frechet differentiable function and $F: X \rightrightarrows 2^Y$ is a set-valued mapping with closed graph. In the present paper, we study the Newton-type method for solving generalized equation $0 \in f(x) + F(x)$. We prove the existence of the sequence generated by the Newton-type method and establish local convergence of the sequence generated by this method for generalized equation.

Key words: Set-valued mapping, **G**eneralized equation, Local convergence, Pseudo-Lipschitz mapping, Newton-type method

1. Introduction

The famous mathematician Robinson [13, 14] was introduced the concept of generalized equation, which was an extension of standard equations with no multi-valued part, as a general tool for describing, analyzing, and solving different problems in a unified manner. These kinds of generalized equation problems have been studied extensively; see for example [5, 13, 14]. Typical examples are systems of inequalities, variational inequalities, linear and non linear complementary problems, systems of nonlinear equations, equilibrium problems etc. Let X and Y be Banach spaces throughout this work unless otherwise stated. Let $f: X \to Y$ be a single-valued function and $F: X \rightrightarrows 2^Y$ be a set-valued mapping with closed graph. In this communication, we are concerned with the problem of approximating a point x^* which satisfies the following generalized equation problem

$$0 \in f(x^*) + F(x^*). \tag{1.1}$$

When the single-valued function involved in (1.1) is differentiable, Newton-like method can be considered to solve this generalized equation, such an approach has been used in many contributions to this subject [2, 3, 9]. To find an approximate solution of $(1,1)$ Dontchev [2] introduced the following classical Newton-type method:

$$0 \in f(x_k) + \nabla f(x_k)(x_{k+1} - x_k) + F(x_{k+1}), \tag{1.2}$$

where $\nabla f(x)$ denotes the Frechet derivative of f at x. It is well known that if $F = \{0\}$, then (1.2) becomes the usual Newton method for solving the equation $f(x) = 0$. If $F = \mathrm{R}^m_+$, the positive orthant in $\mathbb{R}^m$, then (1.1) describes a system of inequalities and (1.2) is a Newton-type method for solving such systems. Finally, the generalized equation (1.1) reduces to the classical variational inequalities:

$$\text{find } x \in \Omega \text{ with } \langle f(x), u - x\rangle \geq 0 \text{ for all } u \in \Omega \tag{1.3}$$

when $F(x) = N_\Omega(x)$ is the normal cone mapping generated by a convex and closed subset $\Omega \subseteq X$. Then the method (1.2) is known version of the Newton method for solving such problems.

Pietrus [9] showed that the sequence generated by the Newton-type method (1.2) converges superlinearly when ∇f is Holder continuous on a neighborhood of $\bar{x}$ and he also proved the stability of this method under mild conditions. Moreover, Geoffroy et al. [6] considered the following method as second degree Taylor polynomial expansion of f for solving the smooth generalized equation (1.1):

$$0 \in f(x_k) + \nabla f(x_k)(x_{k+1} - x_k) + \frac{1}{2}\nabla^2 f(x_k)(x_{k+1} - x_k)^2 + F(x_{k+1}),$$

where $\nabla f(x)$ and $\nabla^2 f(x)$ denote respectively the first and the second Frechet derivative of f at x, and showed the existence of a sequence cubically converging to the solution of (1.1).

Our purpose in this study is to analyze local convergence of the Newton-type method (1.2) for solving generalized equation (1.1). The main tool is the Aubin continuous mapping for set-valued mapping, which was introduced by Aubin in the context of non smooth analysis, and studied by many mathematicians; see for examples [1, 8, 10, 15].

This paper is organized as follows: In Section 2, we recall some necessary notations, notions and preliminary results that will be used in the subsequent sections. In Section 3, we consider the Newton-type method (1.2) for solving the generalized equation (1.1), and establish existence and convergence of the sequence generated by the Newton-type method (1.2). In the last section, we give a summary of the major results of this study.

2. Notations and Preliminaries

Throughout, we assume that X and Y are real or complex Banach spaces. Let $x \in X$ and $r > 0$. The closed ball centered at x with radius r denoted by $B(r, x)$.

The following definition of domain, inverse and graph of a function, distance from a point to a set, excess and pseudo-Lipschitz mapping are taken from [4, 11, 12].

Definition 2.1. Let $F: X \rightrightarrows 2^Y$ be a set-valued mapping. Then the domain of F is denoted by $\operatorname{dom} F$, and is defined by

$$\operatorname{dom} F = \{x \in X : F(x) \neq \phi\}.$$

The inverse of F, denoted by F^{-1}, is defined as

$$F^{-1}(y) = \{x \in X : y \in F(x)\}.$$

While the graph of F, denoted by gph F, is defined by

$$\{(x,y) \in X \times Y : y \in F(x)\}.$$

Definition 2.2. Let X be a Banach space and A be a subset of X . The distance from a point x to a set A is defined by

$$\text{dist}\,(x,A) = \inf\{\|x-y\| : y \in A\}.$$

Definition 2.3. Let X be a Banach space and $A, B \subseteq X$. The excess e from the set A to the set B (also called the Hausdorff semidistance from B to A) is given by

$$e(B,A) = \sup\{\text{dist}(x,A) : x \in B\}.$$

Definition 2.4. Let $F: X \rightrightarrows 2^Y$ be a set-valued mapping. Then F is said to be pseudo-Lipschitz around $(x_0, y_0) \in$ gph F with constant M if there exist positive constants $\alpha, \beta > 0$ such that

$$e\big(F(x_1) \cap B(\beta, y_0), F(x_2)\big) \leq M \parallel x_1 - x_2 \parallel$$

for every $x_1, x_2 \in B(\alpha, x_0)$. When F is single-valued, this corresponds to the usual concept of Lipschitz continuity.

The definition of Lipschitz continuity is equivalent to the definition of Aubin continuity which is given below:

A set-valued mapping $\Gamma: Y \rightrightarrows 2^X$ is said to be Aubin continuous at $(y_0, x_0) \in \text{gph}\Gamma$ with constants a, b and M if for every $y_1, y_2 \in B(b, y_0)$ and for every $x_1 \in \Gamma(y_1) \cap B(a, x_0)$, there exists an $x_2 \in \Gamma(y_2)$ with

$$\parallel x_1 - x_2 \parallel \leq M \parallel y_1 - y_2 \parallel.$$

The constant M is called the modulus of Aubin continuity.

The definition of continuous map, Lipschitz continuous map and Holder continuous map are extracted from [11, 12].

Definition 2.5. Let X and Y be Banach spaces. Then the function $f: \Omega \subseteq X \to Y$ is said to be

(i) Continuous at $\bar{x} \in \Omega$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$\parallel f(x) - f(\bar{x}) \parallel < \varepsilon$ for all points $x \in \Omega$ for which $\parallel x - \bar{x} \parallel < \delta$.

(ii) Lipschitz continuous on an open subset of X if there exists constant c such that

$\parallel f(x) - f(y) \parallel \leq c \parallel x - y \parallel$ for all x and y in the domain of f.

(iii) Holder continuous on an open subset of X if there exists constant c and $p \in (0,1]$ such that
$\parallel f(x) - f(y) \parallel \leq c \parallel x - y \parallel^{p}$ for all x and y in the domain of f. The number p is called the exponent of the Holder condition.

The definition of linear convergence, quadratic convergence and super linear convergence are taken from the book [7].

Definition 2.6. Let $\{x_n\}$ be a sequence which is converges to the number $\bar{x}$. Then the sequence $\{x_n\}$ is said to be

(i) Converges linearly to $\bar{x}$ if there exists a number $0 < c < 1$ such that

$$\| x_{n+1} - \bar{x} \| \leq c \| x_n - \bar{x} \|.$$

(ii) Converges quadratically to $\bar{x}$ if there exists a number $0 < c < 1$ such that

$$\| x_{n+1} - \bar{x} \| \leq c \| x_n - \bar{x} \|^2.$$

(iii) Converges superlinearly to $\bar{x}$ if there exists a number $0 < c < 1$ such that

$$\| x_{n+1} - \bar{x} \| \leq c \| x_n - \bar{x} \|^{\mathrm{p}} , \quad (0 < \mathrm{p} \leq 1) .$$

We end this section with the following lemma, known as fixed point lemma, which can be found in [4].

Lemma 2.1. Let X be a Banach space and $\Phi: \mathrm{X} \rightrightarrows 2^{\mathrm{X}}$ be a set-valued mapping. Let $\bar{x} \in X$; and let r and λ be such that, $0 < \lambda < 1$,

(a) $\operatorname{dist}(\bar{x}, \Phi(\bar{x})) < r(1 - \lambda)$,
(b) $e\big(\Phi(x_1) \cap B(r, \bar{x}), \Phi(x_2)\big) \leq \lambda \| x_1 - x_2 \|$, for all $x_1, x_2 \in B(r, \bar{x})$.

Then Φ has a fixed point in $B(r, \bar{x})$; that is, there exists $x \in B(r, \bar{x})$ such that $x \in \Phi(x)$. If Φ is single-valued, then x is the unique fixed point of Φ in $B(r, \bar{x})$.

3. Convergence Analysis

This section is devoted to study the existence and the convergence of the sequence generated by the Newton-type method (1.2) for the generalized equation (1.1). Let $x \in X$ and define the mapping Q_x by

$$Q_x(\cdot) = f(x) + \nabla f(x)(\cdot - x) + F(\cdot).$$

Moreover, the following equivalence is clear for any $z \in X$ and $y \in Y$:

$$z \in Q_x^{-1}(y) \iff y \in f(x) + \nabla f(x)(z - x) + F(z). \tag{3.1}$$

In particular,

$$x^* \in Q_{x^*}^{-1}(y^*) \text{ for each } (x^*, y^*) \in \operatorname{gph}(f + F).$$

Let $(x^*, y^*) \in \operatorname{gph}(f + F)$ and let $\alpha > 0$, $\beta > 0$. Throughout this section, we assume that $B(\alpha, x^*) \subseteq \Omega \cap \operatorname{dom} F$, where Ω is the neighborhood of x^*.

The following lemma is useful and it plays a vital role in the convergence analysis of the Newton-type method (1.2) for solving the generalized equation (1.1).

Lemma: 3.1. Let X and Y be Banach spaces and let $f: X \to Y$ be a function such that ∇f is Lipschitz continuous on an open subset of X with Lipschitz constant $\kappa > 0$. Then the following statements are equivalent:

(i) The mapping $(f + F)^{-1}$ is pseudo-Lipschitz around (y^*, x^*).

(ii) The mapping $Q_{x^*}^{-1}(\cdot)$ is pseudo-Lipschitz around (y^*, x^*).

Proof: Define a function $g: X \to Y$ by

$$g(x) = -f(x) + f(x^*) + \nabla f(x^*)(x - x^*).$$

To complete the proof of this lemma, according to [4, Corollary 2], we need to show that g is Lipschitz continuous at x^*. To do this, let $\varepsilon > 0$, and take a positive number δ such that $\delta < \frac{\varepsilon}{2\kappa}$. Then, for every $x_1, x_2 \in B(\delta, x^*)$, we have

$$\begin{aligned}
&\| g(x_1) - g(x_2) \| \\
&= \| \left(-f(x_1) + f(x^*) + \nabla f(x^*)(x_1 - x^*)\right) \\
&\qquad - (-f(x_2) + f(x^*) + \nabla f(x^*)(x_2 - x^*)) \| \\
&= \| f(x_2) - f(x_1) - \nabla f(x^*)(x_2 - x_1) \| \\
&\qquad \le \int_0^1 \| \left(\nabla f\left(x_1 + t(x_2 - x_1)\right) - \nabla f(x_1) + \nabla f(x_1) - \nabla f(x^*)\right)(x_2 - x_1) \| \, dt \\
&\le \int_0^1 \| \nabla f\left(x_1 + t(x_2 - x_1)\right) - \nabla f(x_1) \| \| x_2 - x_1 \| \, dt \\
&\qquad + \int_0^1 \| \nabla f(x_1) - \nabla f(x^*) \| \| x_2 - x_1 \| \, dt \\
&\le \kappa \int_0^1 \| t\,(x_2 - x_1) \| \, dt \, \| (x_2 - x_1) \| + \kappa \| x_1 - x^* \| \| x_2 - x_1 \| \\
&= \kappa \left[\frac{t^2}{2} \| x_2 - x_1 \|^2\right]_0^1 + \kappa \| x_1 - x^* \| \| x_2 - x_1 \| \\
&= \left(\frac{\kappa}{2} \| x_2 - x_1 \| + \kappa \| x_1 - x^* \| \right) \| x_2 - x_1 \| \\
&\le \left(\frac{\kappa}{2} \cdot 2\delta + \kappa\delta\right) \| x_2 - x_1 \| = 2\kappa\delta \| x_2 - x_1 \| \\
&\le 2\kappa \cdot \frac{\varepsilon}{2\kappa} \| x_2 - x_1 \| = \varepsilon \| x_2 - x_1 \|.
\end{aligned}$$

Hence g is Lipschitz at x^* and this completes the proof of the lemma.

For our convenience, we define a function $Z_x: X \to Y$ by

$$Z_x(\cdot) = f(x^*) + \nabla f(x^*)(\cdot - x^*) - f(x) - \nabla f(x)(\cdot - x) \tag{3.2}$$

and a set-valued map $\Phi_x: X \rightrightarrows 2^X$ by

$$\Phi_x(\cdot) = {Q_{x^*}}^{-1}[Z_x(\cdot)]. \tag{3.3}$$

Then, for every $x', x'' \in X$, we have

$$\begin{aligned} \| Z_x(x') - Z_x(x'') \| &= \| \big(\nabla f(x^*) - \nabla f(x)\big)(x' - x'') \| \\ &\leq \| \nabla f(x^*) - \nabla f(x) \| \| x' - x'' \| \end{aligned} \tag{3.4}$$

3.1 Linear convergence

If ∇f is continuous, then the following theorem gives us the linear convergence result.

Theorem 3.1 Let x^* be a solution of (1.1). Suppose that ${Q_{x^*}}^{-1}$ is pseudo-Lipschitz around $(0, x^*)$ with Lipschitz constant M and ∇f is continuous on a neighborhood Ω of x^* with constant $L > 0$. Then for every $c > 2ML$ one can find $\delta > 0$ such that for every starting point $x_0 \in B(\delta, x^*)$ there exists a sequence $\{x_k\}$ generated by (1.2), which satisfies

$$\| x_{k+1} - x^* \| \leq c \| x_k - x^* \|. \tag{3.5}$$

Proof: Fix $c > 2ML$ and since ${Q_{x^*}}^{-1}$ is pseudo-Lipschitz with constant M, there exist positive constants a and b such that

$$e\left({Q_{x^*}}^{-1}(y') \cap B(a, x^*), {Q_{x^*}}^{-1}(y'')\right) \leq M \| y' - y'' \|, \text{for all } y', y'' \in B(b, 0). \tag{3.6}$$

Choose $\delta > 0$ and $B(\delta, x^*) \subset \Omega$ so that

$$\delta \leq \min\left(\frac{a}{3}, 1, \frac{\mathrm{b}}{3\mathrm{L}}\right) \quad \text{and} \quad ML \leq \frac{1}{2}. \tag{3.7}$$

Define

$$r_x = 2ML \| x - x^* \|, \qquad \text{for all } x \in X. \tag{3.8}$$

Let $x_0 \in B(\delta, x^*)$, $x_0 \neq x^*$. Utilizing (3.7) in (3.8), we obtain that

$$r_{x_0} \leq 2ML\delta \leq 2 \cdot \frac{1}{2} \cdot \delta \leq \delta. \tag{3.9}$$

Now we will apply Lemma 2.1 to the map Φ_{x_0}, defined in (3.3), with the following specifications:

$$\overline{x} = x^*, \qquad \lambda = \frac{1}{2} \text{ and } r = r_{x_0}.$$

Let us remark that $x^* \in {Q_{x^*}}^{-1}(0) \cap B(\delta, x^*)$. From (3.6) and (3.7), we obtain for $x_0 \in B(\delta, x^*)$, that

$$\begin{aligned} \operatorname{dist}\left(x^*, \Phi_{x_0}(x^*)\right) &\leq e\left({Q_{x^*}}^{-1}(0)\ B(\delta, x^*), \Phi_{x_0}(x^*)\right) \\ &\leq e\big({Q_{x^*}}^{-1}(0) \cap B(a, x^*), {Q_{x^*}}^{-1}\big[Z_{x_0}(x^*)\big]\big). \end{aligned} \tag{3.10}$$

Now, for all $x \in B(\delta, x^*)$, we have that

$$\begin{aligned} \| Z_{x_0}(x) \| &= \| f(x^*) + \nabla f(x^*)(x - x^*) - f(x_0) - \nabla f(x_0)(x - x_0) \| \\ &\leq \| f(x^*) - f(x_0) - \nabla f(x_0)(x^* - x_0) \| + \| \big(\nabla f(x^*) - \nabla f(x_0)\big)(x - x^*) \| \end{aligned}$$

$$\leq \int_0^1 \| \nabla f\big(x_0 + t(x^* - x_0) - \nabla f(x_0)\big)(x^* - x_0) \| dt + \| \nabla f(x^*) - \nabla f(x_0) \| \| x - x^* \|$$
$$\leq L \int_0^1 dt \| x^* - x_0 \| + L \| x - x^* \|$$
$$= L \| x^* - x_0 \| + L \| x - x^* \| \tag{3.11}$$
$$\leq L\delta + L\delta = 2L\delta \leq 2L \cdot \frac{b}{2L} = b.$$

This shows that $Z_{x_0}(x) \in B(b, 0)$ for all $x \in B(\delta, x^*)$. In particular, let $x = x^*$ in (3.11), we get

$$\| Z_{x_0}(x^*) \| \leq L \| x^* - x_0 \|$$
$$\leq L\delta \leq L \cdot \frac{b}{2L} \leq b.$$

Thus, we have that $Z_{x_0}(x^*) \in B(b, 0)$ for $x_0 \in B(\delta, x^*)$.

Now, from the inequality (3.10), we have that

$$\begin{aligned}
\operatorname{dist}\left(x^*, \Phi_{x_0}(x^*)\right) &\leq e\big(Q_{x^*}{}^{-1}(0) \cap B(a, x^*), Q_{x^*}{}^{-1}\big[Z_{x_0}(x^*)\big]\big) \\
&\leq \| Z_{x_0}(x^*) \| \\
&\leq M \| f(x^*) - f(x_0) - \nabla f(x_0)(x^* - x_0) \| \\
&= M \int_0^1 \| \left(\nabla f\big(x_0 + t(x^* - x_0)\big) - \nabla f(x_0)\right)(x^* - x_0) \| dt \\
&\leq ML \textstyle\int_0^1 dt \| x^* - x_0 \| = ML \| x_0 - x^* \| \\
&= \left(1 - \frac{1}{2}\right) 2ML \| x_0 - x^* \| = (1 - \lambda) r_{x_0} \\
&= (1 - \lambda) r .
\end{aligned}$$

That is, the condition (a) of Lemma 2.1 is hold.

Next, we show that condition (b) of Lemma 2.1 is satisfied. Further, using (3.4) and (3.7), for any $x', x'' \in B(r_{x_0}, x^*)$ we obtain that

$$e(\Phi_{x_0}(x') \cap B\big(r_{x_0}, x^*\big), \Phi_{x_0}(x''))$$
$$\leq e\big\{Q_{x^*}{}^{-1}\big[Z_{x_0}(x')\big] \cap B(\delta, x^*), Q_{x^*}{}^{-1}\big[Z_{x_0}(x'')\big]\big\}$$

$$\leq M \| Z_{x_0}(x') - Z_{x_0}(x'') \| \leq M \| \nabla f(x^*) - \nabla f(x_0) \| \| x' - x'' \|$$
$$\leq ML \| x' - x'' \| \leq \frac{1}{2} \| x' - x'' \| = \lambda \| x' - x'' \|.$$

This shows that the condition (b) of Lemma 2.1 is fulfilled. Hence, Φ_{x_0} has a fixed point x_1 in $B(r_{x_0}, x^*)$ such that

$$x_1 \in \Phi_{x_0}(x_1) = Q_{x^*}{}^{-1}\big[Z_{x_0}(x_1)\big]$$
$$\Rightarrow Z_{x_0}(x_1) \in Q_{x^*}(x_1)$$
$$\Rightarrow 0 \in f(x_0) + \nabla f(x_0)(x_1 - x_0) + F(x_1),$$

which means that x_1 is obtained by the method (1.2) for initial point x_0. Thus $x_1 \in B(r_{x_0}, x^*)$ implies that

$$\| x_1 - x^* \| \leq r_{x_0} = 2ML \| x_0 - x^* \| \leq c \| x_0 - x^* \|.$$

Proceeding by induction, and applying the fixed point lemma 2.1 with $\bar{x} = x^*, \lambda = \frac{1}{2}, r = r_{x_k}$

to the map Φ_{x_k} in (3.3), we can deduce the existence of a fixed point $x_{k+1} \in B(\delta, x^*)$ such that

$$x_{k+1} \in \Phi_{x_k}(x_{k+1}) = Q_{x^*}{}^{-1}\left[Z_{x_k}(x_{k+1})\right]$$
$$\Rightarrow Z_{x_k}(x_{k+1}) \in Q_{x^*}(x_{k+1})$$
$$\Rightarrow 0 \in f(x_k) + \nabla f(x_k)(x_{k+1} - x_k) + F(x_{k+1})$$

and hence $\| x_{k+1} - x^* \| \leq c \| x_k - x^* \|$. This shows that (3.5) holds for k. This completes the proof of the theorem.

3.2 Quadratic convergence

In this subsection, we will show in the following theorem that if ∇f is Lipschitz continuous; the sequence generated by method (1.2) converges quadratically.

Theorem 3.2 Let x^* be a solution of (1.1). Suppose that $Q_{x^*}{}^{-1}$ is pseudo-Lipschitz around $(0, x^*)$ with Lipschitz constant M and ∇f is Lipschitz continuous on a neighborhood Ω of x^* with constant $L > 0$. Then for every $c' > \frac{1}{2}ML$ one can find $\delta > 0$ such that for every starting point $x_0 \in B(\delta, x^*)$ there exists a sequence $\{x_k\}$ on a neighborhood Ω of x^* generated by (1.2), which satisfies

$$\| x_{k+1} - x^* \| \leq c' \| x_k - x^* \|^2. \tag{3.12}$$

Proof: Choose $c' > 0$ such that $c' > \frac{1}{2}ML$. Since $Q_{x^*}{}^{-1}$ is pseudo-Lipschitz with constant M, there exist positive constants a and b such that

$$e\left(Q_{x^*}{}^{-1}(y') \cap B(a, x^*), Q_{x^*}{}^{-1}(y'')\right) \leq M \| y' - y'' \|, \text{for all } y', y'' \in B(b, 0). \tag{3.13}$$

Let $\delta > 0$ and $B(\delta, x^*) \subset \Omega$ so that

$$\delta < \min\left\{\frac{a}{2}, \sqrt{\frac{2b}{3L}}, 1, \sqrt{\frac{1}{2ML}}\right\}. \tag{3.14}$$

Define

$$r_x = ML \| x - x^* \|^2, \qquad \text{for all } x \in X. \tag{3.15}$$

Let $x_0 \in B(\delta, x^*)$, $x_0 \neq x^*$. Using (3.14) in (3.15), we get

$$r_{x_0} < ML\delta^2 = ML\delta \cdot \delta \leq \frac{\delta}{2} \leq \delta. \tag{3.16}$$

Now we will apply Lemma 2.1 to the map Φ_{x_0} in (3.3) with the following specifications

$$\bar{x} = x^*, \qquad \lambda = \frac{1}{2}, \qquad r = r_{x_0}.$$

Let us remark that $x^* \in {Q_{x^*}}^{-1}(0) \cap B(\delta, x^*)$.

From (3.13) and (3.14), we obtain, for $x_0 \in B(\delta, x^*)$, that

$$\begin{aligned}\operatorname{dist}\left(x^*, \Phi_{x_0}(x^*)\right) &\leq e\left({Q_{x^*}}^{-1}(0) \cap B(\delta, x^*), \Phi_{x_0}(x^*)\right)\\ &\leq e\left({Q_{x^*}}^{-1}(0) \cap B(a, x^*), {Q_{x^*}}^{-1}\left[Z_{x_0}(x^*)\right]\right). \qquad (3.17)\end{aligned}$$

Now, for all $x \in B(\delta, x^*)$, we have that

$$\begin{aligned}&\| Z_{x_0}(x) \|\\ &= \| f(x^*) + \nabla f(x^*)(x - x^*) - f(x_0) - \nabla f(x_0)(x - x_0) \|\\ &\leq \| f(x^*) - f(x_0) - \nabla f(x_0)(x^* - x_0) \| + \| \left(\nabla f(x^*) - \nabla f(x_0)\right)(x - x^*) \|\\ &\leq \frac{L}{2} \| x^* - x_0 \|^2 + L \| x^* - x_0 \| \| x - x^* \| \qquad (3.18)\\ &\leq \frac{L}{2}\delta^2 + L\delta^2 = \frac{3}{2}L\delta^2\\ &\leq \frac{3L}{2} \cdot \frac{2b}{3L} = b.\end{aligned}$$

So, we have $Z_{x_0}(x) \in B(b, 0)$ for all $x \in B(\delta, x^*)$. In particular, let $x = x^*$ in (3.18), we get

$$\begin{aligned}\| Z_{x_0}(x^*) \| &\leq \frac{L}{2} \| x^* - x_0 \|^2 \leq \frac{L}{2}\delta^2\\ &\leq \frac{L}{2} \cdot \frac{2b}{3L} \leq b\end{aligned}$$

Thus, we have $Z_{x_0}(x^*) \in B(b, 0)$.

Now, from the inequality (3.17), we get

$$\begin{aligned}\operatorname{dist}\left(x^*, \Phi_{x_0}(x^*)\right) &\leq e\left({Q_{x^*}}^{-1}(0) \cap B(a, x^*), {Q_{x^*}}^{-1}\left[Z_{x_0}(x^*)\right]\right)\\ &\leq \| Z_{x_0}(x^*) \|\\ &\leq M \| f(x^*) - f(x_0) - \nabla f(x_0)(x^* - x_0) \|\\ &\leq M \int_0^1 \| \left(\nabla f\left(x_0 + t(x^* - x_0)\right) - \nabla f(x_0)\right)(x^* - x_0) \| \, dt\\ &\leq \frac{1}{2} ML \| x_0 - x^* \|^2\\ &= \left(1 - \frac{1}{2}\right) ML \| x_0 - x^* \|^2 = (1 - \lambda) r_{x_0}\\ &= (1 - \lambda) r\,.\end{aligned}$$

This means that condition (a) of Lemma 2.1 is hold.

Next, we show that condition (b) of Lemma 2.1 is also satisfied. To do this, let $x', x'' \in B(r_{x_0}, x^*)$. Then using (3.4) and (3.14), we obtain that

$$
\begin{aligned}
& e(\Phi_{x_0}(x') \cap B(r_{x_0}, x^*), \Phi_{x_0}(x'')) \\
& \leq e\{Q_{x^*}{}^{-1}[Z_{x_0}(x')] \cap B(\delta, x^*), Q_{x^*}{}^{-1}[Z_{x_0}(x'')]\} \\
& \leq M \parallel Z_{x_0}(x') - Z_{x_0}(x'') \parallel \\
& \leq M \parallel \nabla f(x^*) - \nabla f(x_0) \parallel \parallel x' - x'' \parallel \\
& \leq ML \parallel x^* - x_0 \parallel^2 \parallel x' - x'' \parallel \\
& \leq ML\delta^2 \parallel x' - x'' \parallel \leq \frac{1}{2} \parallel x' - x'' \parallel \\
& = \lambda \parallel x' - x'' \parallel;
\end{aligned}
$$

that is, condition (b) of Lemma 2.1 is fulfilled. Hence, Φ_{x_0} has a fixed point x_1 in $B(r_{x_0}, x^*)$ such that

$$
\begin{aligned}
& x_1 \in \Phi_{x_0}(x_1) = Q_{x^*}{}^{-1}[Z_{x_0}(x_1)] \\
& \Rightarrow Z_{x_0}(x_1) \in Q_{x^*}(x_1) \\
& \Rightarrow 0 \in f(x_0) + \nabla f(x_0)(x_1 - x_0) + F(x_1),
\end{aligned}
$$

which means that x_1 is obtained by the method (1.2) for initial point x_0. Thus $x_1 \in B(r_{x_0}, x^*)$ implies that

$$\parallel x_1 - x^* \parallel \leq r_{x_0} = ML \parallel x_0 - x^* \parallel^2 \leq c' \parallel x_0 - x^* \parallel^2$$

Proceeding by induction, and applying the fixed point lemma with $\bar{x} = x^*, \lambda = \frac{1}{2}, r = r_{x_k}$ to the map Φ_{x_k} in (3.3), we can deduce the existence of a fixed point $x_{k+1} \in B(\delta, x^*)$ such that

$$
\begin{aligned}
& x_{k+1} \in \Phi_{x_k}(x_{k+1}) = Q_{x^*}{}^{-1}[Z_{x_k}(x_{k+1})] \\
& \Rightarrow Z_{x_k}(x_{k+1}) \in Q_{x^*}(x_{k+1}) \\
& \Rightarrow 0 \in f(x_k) + \nabla f(x_k)(x_{k+1} - x_k) + F(x_{k+1})
\end{aligned}
$$

and hence $\parallel x_{k+1} - x^* \parallel \leq c' \parallel x_k - x^* \parallel^2$. This shows that (3.12) holds for k. This completes the proof of the theorem.

3.3 Super linear convergence

If ∇f is Holder continuous, the following theorem gives us the super linear convergence of the method (1.2).

Theorem 3.3 Let x^* be a solution of (1.1). Let $p \in [0, 1]$ and suppose that $Q_{x^*}{}^{-1}$ is pseudo-Lipschitz around $(0, x^*)$ with Lipschitz constant M and ∇f is Holder continuous on a neighborhood Ω of x^* with constant $L > 0$. Then for every $c'' > \frac{ML}{p+1}$ one can find $\delta > 0$ such that for every starting point $x_0 \in B(\delta, x^*)$ there exists a sequence $\{x_k\}$ on a neighborhood Ω of x^* generated by (1.2), which satisfies

$$\parallel x_{k+1} - x^* \parallel \leq c'' \parallel x_k - x^* \parallel^{p+1}. \tag{3.19}$$

Proof: Fix $c'' > \frac{ML}{p+1}$ and since $Q_{x^*}{}^{-1}$ is pseudo-Lipschitz with constant M, there exist positive constants a and b such that

$$e\left(Q_{x^*}{}^{-1}(y') \cap B(a, x^*), Q_{x^*}{}^{-1}(y'')\right) \leq M \parallel y' - y'' \parallel, \text{for all } y', y'' \in B(b, 0). \quad (3.20)$$

Choose $\delta > 0$ and $B(\delta, x^*) \subset \Omega$ so that

$$\delta \leq \min\left\{\left(\frac{b(p+1)}{L(p+2)}\right)^{\frac{1}{p+1}}, \left(\frac{p}{ML(p+1)}\right)^{\frac{1}{p}}\right\}. \quad (3.21)$$

Define

$$r_x = ML \parallel x - x^* \parallel^{p+1}. \quad (3.22)$$

Let $x_0 \in B(\delta, x^*)$, $x_0 \neq x^*$. Utilizing (3.21) in (3.22), we get

$$r_{x_0} \leq ML\delta^{p+1} = ML\delta^p \cdot \delta \leq \frac{p}{p+1}\delta \leq \delta. \quad (3.23)$$

Now we will apply Lemma 2.1 to the map Φ_{x_0} in (3.3) with the following specifications:

$$\bar{x} = x^*, \qquad \lambda = \frac{p}{p+1}, \qquad r = r_{x_0}.$$

Let us remark that $x^* \in Q_{x^*}{}^{-1}(0) \cap B(\delta, x^*)$. From (3.20) and (3.21), we obtain, for $x_0 \in B(\delta, x^*)$, that

$$\begin{aligned} \operatorname{dist}\left(x^*, \Phi_{x_0}(x^*)\right) &\leq e\left(Q_{x^*}{}^{-1}(0) \cap B(\delta, x^*), \Phi_{x_0}(x^*)\right) \\ &\leq e\left(Q_{x^*}{}^{-1}(0) \cap B(a, x^*), Q_{x^*}{}^{-1}\left[Z_{x_0}(x^*)\right]\right). \quad (3.24) \end{aligned}$$

Now, for all $x \in B(\delta, x^*)$, we have that

$$\begin{aligned} &\parallel z_{x_0}(x) \parallel \\ &= \parallel f(x^*) + \nabla f(x^*)(x - x^*) - f(x_0) - \nabla f(x_0)(x - x_0) \parallel \\ &= \parallel f(x^*) - f(x_0) - \nabla f(x_0)(x^* - x_0) \parallel + \parallel \left(\nabla f(x^*) - \nabla f(x_0)\right)(x - x^*) \parallel \\ &\leq \int_0^1 \parallel \left(\nabla f\left(x_0 + t(x^* - x_0)\right) - \nabla f(x_0)\right)(x^* - x_0) \parallel dt \\ &\quad + \parallel \nabla f(x^*) - \nabla f(x_0) \parallel \parallel x - x^* \parallel \\ &\leq \int_0^1 \parallel \nabla f\left(x_0 + t(x^* - x_0)\right) - \nabla f(x_0) \parallel dt \parallel x^* - x_0 \parallel + L \parallel x^* - x_0 \parallel^p \parallel x - x^* \parallel \\ &\leq L \int_0^1 \parallel t(x^* - x_0) \parallel^p dt \parallel x^* - x_0 \parallel + L \parallel x^* - x_0 \parallel^p \parallel x - x^* \parallel \\ &= \frac{L}{p+1} \parallel x^* - x_0 \parallel^{p+1} + L \parallel x^* - x_0 \parallel^p \parallel x - x^* \parallel \quad (3.25) \\ &\leq \frac{L}{p+1}\delta^{p+1} + L\delta^{p+1} = \frac{L(p+2)}{p+1}\delta^{p+1}. \end{aligned}$$

Then for $L(p+2)\delta^{p+1} \leq b(p+1)$ in (3.21), we have

$$\| z_{x_0}(x) \| \leq b.$$

So we have $Z_{x_0}(x) \in B(b,0)$ for all $x \in B(\delta, x^*)$.

In particular, let $x = x^*$ in (3.25), we get

$$\begin{aligned}\| z_{x_0}(x^*) \| &\leq \frac{L}{p+1} \| x^* - x_0 \|^{p+1} \\ &\leq \frac{L}{p+1}\delta^{p+1} \leq \frac{L}{p+1} \cdot \frac{b(p+1)}{L(p+2)} \leq b.\end{aligned}$$

Then by (3.21), we have that $Z_{x_0}(x^*) \in B(b,0)$.

Now, from the inequality (3.24), we have that

$$\begin{aligned}\operatorname{dist}\left(x^*, \Phi_{x_0}(x^*)\right) &\leq e\left({Q_{x^*}}^{-1}(0) \cap B(a,x^*), {Q_{x^*}}^{-1}\left[Z_{x_0}(x^*)\right]\right) \\ &\leq \| Z_{x_0}(x^*) \| \\ &\leq M \| f(x^*) - f(x_0) - \nabla f(x_0)(x^* - x_0) \| \\ &= M \int_0^1 \| \left(\nabla f\left(x_0 + t(x^* - x_0)\right) - \nabla f(x_0)\right)(x^* - x_0) \| \, dt \\ &\leq M \int_0^1 \| \nabla f\left(x_0 + t(x^* - x_0)\right) - \nabla f(x_0) \| \, dt \, \| x^* - x_0 \| \\ &\leq ML \int_0^1 \| t(x^* - x_0) \|^p \, dt \, \| x^* - x_0 \| \\ &= ML \left[\frac{t^{p+1}}{p+1}\right]_0^1 \| x^* - x_0 \|^{p+1} \\ &= \frac{ML}{p+1} \| x^* - x_0 \|^{p+1} \\ &= \left(1 - \frac{p}{p+1}\right) ML \| x^* - x_0 \|^{p+1} = (1-\lambda) r_{x_0} \\ &= (1-\lambda) r \, ;\end{aligned}$$

This shows that the condition (a) of Lemma 2.1 is satisfied.

Next, we show that condition (b) of Lemma 2.1 also holds. To complete this, let $x', x'' \in B(r_{x_0}, x^*)$. Now, using (3.4) and (3.21), we obtain that

$$\begin{aligned}& e(\Phi_{x_0}(x') \cap B\left(r_{x_0}, x^*\right), \Phi_{x_0}(x'')) \\ & \leq e\left\{{Q_{x^*}}^{-1}\left[Z_{x_0}(x')\right] \cap B(\delta, x^*), {Q_{x^*}}^{-1}\left[Z_{x_0}(x'')\right]\right\} \\ & \leq M \| Z_{x_0}(x') - Z_{x_0}(x'') \| \leq M \| \nabla f(x^*) - \nabla f(x_0) \| \| x' - x'' \| \\ & \leq ML \| x^* - x_0 \|^p \| x' - x'' \| \leq ML\delta^p \| x' - x'' \|\end{aligned}$$

$$\leq \frac{p}{p+1} \parallel x' - x'' \parallel \; = \lambda \parallel x' - x'' \parallel ;$$

This indicates that the condition (b) of Lemma 2.1 is fulfilled. Hence, Φ_{x_0} has a fixed point x_1 in $B(r_{x_0}, x^*)$ such that

$$x_1 \in \Phi_{x_0}(x_1) = {Q_{x^*}}^{-1}\big[Z_{x_0}(x_1)\big]$$
$$\Rightarrow Z_{x_0}(x_1) \in Q_{x^*}(x_1)$$
$$\Rightarrow 0 \in f(x_0) + \nabla f(x_0)(x_1 - x_0) + F(x_1),$$

which means that x_1 is obtained by the method (1.2) for initial point x_0. Thus $x_1 \in B(r_{x_0}, x^*)$ implies that

$$\parallel x_1 - x^* \parallel \leq r_{x_0} = ML \parallel x_0 - x^* \parallel^{p+1} \leq c'' \parallel x_0 - x^* \parallel^{p+1}.$$

Proceeding by induction, and applying the fixed point lemma with $\bar{x} = x^*, \lambda = \frac{p}{p+1}, r = r_{x_k}$ to the map Φ_{x_k} in (3.3), we can deduce the existence of a fixed point $x_{k+1} \in B(\delta, x^*)$ such that

$$x_{k+1} \in \Phi_{x_k}(x_{k+1}) = {Q_{x^*}}^{-1}\big[Z_{x_k}(x_{k+1})\big]$$
$$\Rightarrow Z_{x_k}(x_{k+1}) \in Q_{x^*}(x_{k+1})$$
$$\Rightarrow 0 \in f(x_k) + \nabla f(x_k)(x_{k+1} - x_k) + F(x_{k+1})$$

and hence

$$\parallel x_{k+1} - x^* \parallel \leq c'' \parallel x_k - x^* \parallel^{p+1}$$

This shows that (3.19) holds for k. This completes the proof of the theorem.

4. Concluding Remarks

We have established the local convergence of the Newton-type method under the assumptions that ${Q_{x^*}}^{-1}(\cdot)$ is pseudo-Lipschitz and ∇f is continuous, Lipschitz-continuous and Holder continuous. We have shown that if ∇f is continuous, Lipschitz continuous and Holder continuous then the Newton-type method is linearly convergent, quadratically convergent and super linearly convergent respectively. This investigation improves and extends the results corresponding to [2].

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