ENDOMORPHISM RINGS ARE CENTRALIZER NEAR-RINGS

Md. Rezaul Islam\textsuperscript{1} and Satrajit Kumar Saha\textsuperscript{2}
\textsuperscript{1}Dhaka Cantonment Girls’ Public School and College, Dhaka.
\textsuperscript{2}Department of Mathematics, Jahangirnagar University, Savar, Dhaka.
\textsuperscript{1}E-mail: rezaadhimoni@gmail.com

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ABSTRACT

For a finite ring $R$ with identity and a finite unital $R$-module $V$ the set $C(R; V) = \{ f : V \to V : f(\alpha v) = \alpha f(v) \ \text{for all} \ \alpha \in R, v \in V \}$ is the centralizer near-ring determined by $R$ and $V$. Rings $R$ for which $C(R; V)$ is a ring for every $R$-module $V$, are characterized. Conditions are given under which $C(R; V)$ is a semisimple centralizer near ring. It's shown that $C(R; V)$ is a semisimple centralizer near ring then $\text{End}_R(V) = C(R; V)$.

Keywords: Centralizer Near-ring, Semisimple Centralizer Near-ring, $C(R, V)$ invariant

1. Introduction.

Let $G$ be a group and $T$ a semigroup of endomorphisms of $G$. Then $C(T; G) = \{ f : G \to G : f(0) = 0 \ \text{and} \ f(\lambda a) = \lambda f(a) \ \text{for all} \ \lambda \in T, a \in G \}$ is a near-ring under the operations of addition and composition of functions, and is called the centralizer near-ring determined by $T$ and $G$. It has been shown by Betsch [1] that $N$ is a finite simple near-ring with identity if and only if there exists a finite group $G$ and a fixed point free group of automorphism $T$ of $G$ such that $N \cong C(T; G)$. The structure of $C(T; G)$ for various $G$ and $T$ has been investigated by Maxson and Smith [3], [4], [5].

Throughout this paper $R$ will denote a finite ring with 1 and $V$ a finite unital $R$-module. The corresponding centralizer near-ring is $C(R; V) = \{ f : V \to V : f(\alpha v) = r f(v) \ \text{for all} \ r \in R, \ v \in V \}$. In dealing with $C(R; V)$ we may assume, without loss of generality, that $V$ is a faithful $R$-module, because $V$ is a faithful $\overline{R}$-module where $\overline{R} = R/\text{Ann}(V)$, and we have $C(R; V) = C(\overline{R}, V)$.

It is the goal of this paper to consider the following questions which arise naturally from the above remarks.

A. Which finite rings $R$ have the property that $C(R; V)$ is a ring for every $R$-module $V$?

B. If $C(R; V)$ is a semisimple ring when is $C(R; V) = \text{End}_R(V)$?

C. Which semisimple near-rings have the form $C(R; V)$ for some pair $(R, V)$?

Subsequently we will answer question A. We also show that if $C(R; V)$ is a semisimple ring then one always has $C(R; V) = \text{End}_R(V)$. Moreover if $C(R; V)$ is semisimple then information about the structure of the simple components is obtained, giving a partial
answer to question C. Subsequently we will show that if $C(R; V)$ is a semisimple ring then $End_R(V) = C(R; V)$.

2. Semisimple Centralizer Near-ring

At first we will define semisimple centralizer near-ring. Then some characteristics or properties of semisimple centralizer near-ring will be established.

**Definition 2.1. Semisimple Centralizer Near-rings** [2] Let $C(R; V)$ be semisimple. Then the center of $C(R; V)$ cannot contain nonzero nilpotent elements. Hence the center of $R$ cannot contain nilpotent elements so the center of $R$ is a direct sum of fields. Thus if $n$ is the characteristic of $R$, we have $n = p_1 ... p_s$ where $p_i$'s are distinct primes. But this implies that $R = R_1 \oplus ... \oplus R_s$ where $R_i$ has characteristic $p_i$. Because it has characteristic $p_i$, $R_i$ is an algebra over the field $GF(p_i)$ and so the Wedderburn principal theorem [7, p. 164] holds for $R$. Consequently $R = \sum_{y} S_y + N$ where each $S_y$ is a simple ring and $N$ is a nilpotent ideal of $R$.

**Proposition 2.1.** [2]. Let $R$ be a finite semisimple ring and let $V$ be a finite $R$-module. Then $C(R; V)$ is a semisimple near-ring.

**Proof:** We have $R = S_i \oplus ... \oplus S_i$, where each $S_i$ is a simple ring. Let $e_i$ denote the identity of $S_i$. If $V_i = \{v \in V : e_i v = v\}$ then $V = V_1 \oplus ... \oplus V_s$, and $f(V_i) \subseteq V_i$ for each $f \in C(R; V)$. Further, if $f_i$ denotes the restriction of $f$ to $V_i$ then the map $\phi : C(R; V) \rightarrow C(S_1; V_1) \oplus ... \oplus C(S_s; V_s)$ given by $\phi(f) = \langle f_1, ..., f_s \rangle$ is a near-ring homomorphism. The map is onto, for if $\langle f_1, ..., f_s \rangle$ is in $C(S_1; V_1) \oplus ... \oplus C(S_s; V_s)$ extend each $f_i$ to all of $V$ by $\tilde{f}_i (v_1 + ... + v_i) = f_i(v_i)$. Then $f = \sum \tilde{f}_i$ is an element of $C(R; V)$ such that $\phi(f) = \langle f_1, ..., f_s \rangle$. To show that $\phi$ is one-to-one we note that $e_i f(v_1 + ... + v_i) = f(e_i v_i) = f(v_i), i = 1, ..., t$ implies $f(v_1 + ... + v_i) = f(v_1) + ... + f(v_i) = f_1(v_1) + ... + f_s(v_s)$. Hence if $\phi(f) = 0$ then $f = 0$. Therefore $\phi$ is an isomorphism and from Theorem 1 of [6] each $C(S_i; V_i)$ is a simple near-ring.

**Proposition 2.2.** If $C(R; V)$ is a semisimple near-ring for every $R$-module $V$ then in particular $C(R; R)$ is semisimple. But $C(R; R)$ is anti-isomorphic to $R$ so $R$ is a semisimple ring.

**Proof:** If $R = S_i \oplus ... \oplus S_i$, simple and not a field, or $S_i$ is a field and $\dim_{S_i}(V_i) = 1$, we have $C(R; V)$ is a semisimple ring. Moreover, in this setting, $C(R; V) = End_R(V)$.

**Theorem 2.1.** $C(R; V)$ is a semisimple ring which is not a ring.

**Proof:** We will prove this theorem with the help of an example. Let $R = \bar{R} \oplus F$ where $F = GF(2)$ and $\bar{R}$ is the simple ring of $2 \times 2$ matrices over $GF(2)$. Let
\(V_i = \left\{ \left\{ \begin{array}{l} x \\ y \end{array} \right\} : x, y \in F \right\} \), \(i = 1, 2\), and let \(R\) act on \(V = V_1 \oplus V_2\) componentwise. Then \(C(R; V) \cong C(R; V_1) \oplus C(F; V_2)\) where \(C(R; V_1)\) is a simple ring while \(C(F; V_2)\) is a simple near-ring which is not a ring. Hence \(C(R; V)\) is semisimple and not a ring.

**Theorem 2.2.** When \(C(R; V)\) is semisimple centralizer near-ring then \(\text{End}_R(V) = C(R; V)\).

**Proof:** As we have seen \(R = S_i \oplus \ldots \oplus S_i + N\) where each \(S_i\) is simple and \(N\) is a nilpotent ideal of \(R\). We may assume \(N \neq (0)\); otherwise \(R\) is semisimple and the previous result applies.

Assume \(t = 1\), i.e. \(R = S_i + N\). From the proof of Lemma 1 of \([6]\) it follows that \(C(R; V)\) contains a function \(f\) such that \(g_1 f g_2 f = 0\) for all \(g_1, g_2 \in C(R; V)\). Hence \(C(R; V)\) contains a nilpotent \(C(R; F)\)-subgroup and is not semisimple. So we may assume \(t > 1\).

Let \(\epsilon_i\) denote the identity for \(S_i\). Then \(V = V_1 \oplus \ldots \oplus V_t\) where \(V_j = \{ v \in V : e_j v = v \}\). Also for \(i, j = 1, 2, \ldots, t\) let \(N_{ij} = e_i N e_j\). Then \(N = \sum N_{ij}\). For \(i = 1, \ldots, t\) let \(B_i = \{ w \in V : w_j = n_{ij} v_j \text{ for some } j \neq i, n_{ij} \in N_{ij}, v_j \in V_j \}\), and let \(W\) denote the subgroup of \(V\) generated by \(B_i \cup B_2 \cup \ldots \cup B_t\). Finally let \(W_i = \{ w \in V : f(w + v) = f(w) + f(v)\text{ for all } v \in V, f \in C(R; V) \}\). Therefore it is clear that \(\text{End}_R(V) = C(R; V)\).

**Corollary 2.1.** If \(R\) is not semisimple then at least one \(A_i\) must be a ring.

**Proof:** Since \(C(R; V)\) is semisimple then \(R = S_i \oplus \ldots \oplus S_i + N\) where \(N = \text{rad} R\) and each \(S_i\) is simple with identity \(e_i\). As before let \(N_{ij} = e_i N e_j\) and let \(W\) be the \(R\)-submodule of \(V\) as in Lemma 1. If \(W = (0)\) then \(N_{ij} V_j = (0)\) for each \(i \neq j\) where \(V_j\) is the 1-space for \(e_j\).

This means each \(V_i\) is an \(R\)-module as well as \(C(R; V)\)-invariant. Hence

\[C(R; V) \cong C(R_i; V_i) \oplus \ldots \oplus C(R_k; V_k)\]

where \(R_i = S_i + N_{ii}\). Since \(C(R; V)\) is semisimple each \(C(R_i; V_i)\) is semisimple \([8,\text{p.} 146]\).

We show now that if \(N_{ii} \neq (0)\) then \(C(R_i; V_i)\) cannot be semisimple.

Suppose \(N_{ii}^{-1} = (0)\) but \(N_{ii}^{-1} \neq (0)\). Let \(W = \ker N_{ii}^{-1} = \{ v \in V_i : nv = 0 \text{ for all } n \in N_{ii}^{-1} \}\), a proper subgroup of \(V_i\), an \(S_i\)-submodule, and \(C(R_i; V_i)\)-invariant. As \(S_i\)-module \(V_i\) is completely reducible so \(V_i = W_i \oplus W_2\), an \(S_i\)-module direct sum. As constructed in the proof of Lemma 1 of \([6]\) there exists a nonzero \(f \in C(R_i; V_i)\) such that \(f(V_i) \subseteq W_i\) and \(f(W_i) = (0)\). Let \(I = \{ f \in C(R_i; V_i) : f(V_i) \subseteq W_i \text{ and } f(W_i) = (0) \}\). Then \(I\) is a nilpotent
$C(R_i;V_i)$-subgroup ($I^2 = (0)$) and hence $C(R_i;V_i)$ is not semisimple. So each $N_{ii} = (0)$ and since $N_{ij}V = (0), N_{ij} = (0)$ if $i \neq j$. Thus $R$ is semisimple.

So we may assume $W \neq (0)$. Since $W$ is $C(R; V)$-invariant the map $f \rightarrow f/W$ is a homomorphism of $C(R; V)$ into the ring $End_R(W)$. Hence a non-trivial homomorphic image of $C(R; V)$ is a ring and this implies at least one simple component of $C(R; V)$ is a ring [8, p. 55].

3. Simple Centralizer Near-ring

In this section we will discuss simple centralizer near-ring and ring with their characteristics or properties.

**Theorem 3.1.** $C(R; V)$ is a ring when $End_R(V) \neq C(R; V)$.

**Proof:** We will prove this theorem with the help of an example. Let $R$ be the ring consisting of the $3 \times 3$ matrices of the form

$$
\begin{bmatrix}
a & b & c \\
0 & a & 0 \\
0 & 0 & a
\end{bmatrix}, \quad a, b, c \in GF(2).
$$

Let

$$
V = \begin{bmatrix} x \\ y \\ z \end{bmatrix} : x, y, z \in GF(2).
$$

A calculation shows that $End_R(V) = R$. Another calculation gives $f(Rv) \subseteq Rv$ for each $f \in C(R; V)$ and for each $v \in V$. From this it follows that $C(R; V)$ is a ring since if $v \in V$ then

$$
f(g + h)v = f((g + h)v) = f(gv + hv) = f(rv + rv) = (r_1 + r_2)f(v) = (fg + fh)v.
$$

Let $\{e_1, e_2, e_3\}$ be the standard basis for the vector space $V$ over $GF(2)$. Then $V = R(e_1 + e_2 + e_3) \cup Re_1 \cup Re_3$ and the relation $f(e_1 + e_2 + e_3) = f(e_2) = f(e_3) = e_1$ determines a function in $C(R; V)$. But $f$ is not in $End_R(V)$ since $f(e_2 + e_3) \neq f(e_2) + f(e_3)$. Hence $End_R(V) \neq C(R; V)$.

**Lemma 3.1.** [2]. $W$ is an $R$-submodule of $V$, $W_L$ is a subgroup of $V$ and $W \subseteq W_L$, where

$$
W_k = \{w \in V : f(w + v) = f(w) + f(v) \text{ for all } v \in V, f \in C(R; V)\}.
$$

**Proof:** An element of $W$ has the form $w = \sum n_{ij}v_{ij}$ with $i \neq j$. For $n_{kl} \in N_{kl}$ and $n_{ij}v_{ij} \in B_j$ we have $n_{kl}n_{ij}v_{ij} \in B_k$ if $k \neq j$ and $n_{kl}n_{ij}v_{ij} = n_{kl}(n_{ij}v_{ij}) \in B_k$ if $k = j$. In this manner it is
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seen that \( NW \subseteq W. \) Also if \( s \in S_i \oplus \cdots \oplus S_l \) then \( sn_j v_j = (sn_j) v_j \in B_i \) since \( sn_j \in N_j. \) Hence \( SW \subseteq W \) and \( W \) is an \( R \)-submodule of \( V. \)

The second part of the lemma is straight forward and is not discussed. To prove the last part it suffices to show that \( B_i \subseteq W_L \) for each \( i. \) To this end let \( v_i = n_j v_j \in B_i, f \in C(R; V). \) For \( k \neq i \) we have \( f(v_i + v_k) = f(v_i) + f(v_k). \) For \( v_i' \in V_i, \)

\[
f(v_i + v_i') = f(n_j v_i + v_i') = f((n_j + e_j)(v_i + v_i'))
\]

\[
= (n_j + e_j)f(v_i + v_i') = (n_j + e_j)[f(v_i) + f(v_i')] = f(v_i) + f(v_i').
\]

With this it is easy to see that \( f(v_i + v) = f(v_i) + f(v) \) for all \( v \in V, \) as desired.

**Lemma 3.2.** \( C(R; V) \) would not be simple but it is a ring.

**Proof:** From Lemma 3.1, every \( f \in C(R; V) \) is linear on \( W \) and moreover \( f(W) \subseteq W. \) Suppose now that \( C(R; V) \) is simple. Then the map \( f \to \frac{f}{W} \) is an imbedding of \( C(R; V) \) into \( \text{End}_R(W). \) Also \( W \neq (0), \) for otherwise \( n_j V_j = (0) \) for each \( i \neq j \) and so each \( V_i \) is an \( R \)-module and \( C(R; V) \)-invariant. Hence \( C(R; V) \) would not be simple. Thus \( W \neq 0 \) and \( C(R; V) \) is a ring.

4. Conclusion

Starting with the definition of centralizer near-ring throughout the paper we have discussed semisimple and simple centralizer near-rings with their various characteristics or properties. We have established Proposition 2.2, Theorem 2.1, Theorem 2.2, Corollary 2.1, Theorem 3.1 and Lemma 3.2.

**REFERENCES**


