COMMUTATIVELY OF PRIME AND SEMIPRIME
Γ-RINGS WITH SYMMETRIC BI-DERIVATIONS

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ABSTRACT

Let $M$ be a $\Gamma$-ring and let $D: M \times M \rightarrow M$ be a symmetric bi-derivation with the trace $d: M \rightarrow M$ denoted by $d(x) = D(x, x)$ for all $x \in M$. The objective of this paper is to prove some results concerning symmetric bi-derivation on prime and semiprime $\Gamma$-rings. If $M$ is a 2-torsion free prime $\Gamma$-ring and $D \neq 0$ be a symmetric bi-derivation with the trace $d$ having the property $d(x)\alpha - xd(x) = 0$ for all $x \in M$ and $\alpha \in \Gamma$, then $M$ is commutative. We also prove another result in $\Gamma$-rings setting analogous to that of Posner for prime rings.

Keywords: $\Gamma$-ring, derivation, bi-derivation, commutativity

1. Introduction and Preliminaries


In this paper, we extend some results of Vukman [10] to prime and semiprime $\Gamma$-rings. Our results are quiet different from the results obtained in [9].

Let $M$ and $\Gamma$ be additive abelian groups. If there exists a mapping $(x, \alpha, y) \rightarrow x\alpha y$ of $M \times \Gamma \times M \rightarrow M$ which satisfies the conditions:

(i) $x\alpha y \in M,$
(ii) $(x + y)\alpha z = x\alpha z + y\alpha z,$
(iii) $(x\alpha y)\beta z = x\alpha(y\beta z)$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma,$

then $M$ is called a $\Gamma$-ring in the sense of Barnes [1]. Throughout this paper $M$ denotes a $\Gamma$-ring with center $Z(M).$ For any $x, y \in M, \alpha \in \Gamma,$ the symbol $[x, y]_\alpha$ (resp. $(x, y)_\alpha$) will denote the commutator $x\alpha y - y\alpha x$ (resp. the anti-commutator $x\alpha y + y\alpha x$). A $\Gamma$-ring $M$ is called commutative if $[x, y]_\alpha = 0$ for all $x, y \in M, \alpha \in \Gamma.$ We know that

$[x\beta y, z]_\alpha = [x, z]_\alpha \beta y + x\beta[y, z]_\alpha + x[\beta, \alpha]_\beta y$
and
\[ [x, y]z]_a = yβ[x, z]_a + [x, y]_aβz + yβ, α]z. \]
We make the assumption (*) \( xβzαy = xαzβy \) for all \( x, y, z \in M, α, β \in Γ \). Using this assumption the basic commutator identities reduce to
\[ [xβy, z]_a = [x, z]_aβy + xβ[y, z]_a \]
\[ [x, yβz]_a = yβ[x, z]_a + [x, y]_aβz. \]

Recall that a Γ-ring is prime if \( xΓMΓy = 0 \) implies that \( x = 0 \) or \( y = 0 \), and is semiprime if \( xΓMΓx = 0 \) implies \( x = 0 \). An additive mapping \( d: M \to M \) is called a derivation if \( d(xαy) = d(x)αy + xαd(y) \) holds for all \( x, y \in M, α \in Γ \). A derivation \( d \) is inner if there exists \( a \in M \) such that \( d(x) = [a, x]_a \) holds for all \( x \in M, α \in Γ \). The mapping \( B: M \times M \to M \) is said to be symmetric if \( B(x, y) = B(y, x) \) holds for all \( x, y \in M \). A mapping \( f: M \to M \) defined by \( f(x) = B(x, x) \), where \( B: M \times M \to M \) is a symmetric mapping, is called the trace of \( B \).

In case \( B: M \times M \to M \) is a symmetric mapping which is also bi-additive (i.e., additive in both arguments), the trace of \( B \) satisfies the relation \( f(x + y) = f(x) + f(y) + 2B(x, y) \), for all \( x, y \in M \). We shall use also the fact that the trace of a symmetric bi-additive mapping is an even function. A symmetric bi-additive mapping \( D: M \times M \to M \) is called a symmetric bi-derivation if \( D(xαy, z) = D(x, z)αy + xaD(y, z) \) is fulfilled for all \( x, y, z \in M, α \in Γ \). Obviously, in this case also the relation \( D(x, yαz) = D(x, y)αz + yαD(x, z) \) for all \( x, y, z \in M, α \in Γ \), holds.

2. Bi-derivations on Γ-rings

We shall need the following well-known and frequently used lemmas.

**Lemma 2.1.** ([2, Lemma 3.2]) Let \( d: M \to M \) be a derivation, where \( M \) is a prime Γ-ring. Suppose that either (i) \( dΓd(x) = 0 \), for all \( x \in M \) or (ii) \( dΓx = 0 \), for all \( x \in M \) holds. Then we have (i) \( a = 0 \) or (ii) \( d = 0 \).

**Lemma 2.2.** ([7, Lemma 3]) Let \( M \) be a 2-torsion free prime Γ-ring and let \( a, b \in M \) be fixed elements. If \( aαβb + bαβa = 0 \) is fulfilled for all \( x, y \in M, α, β \in Γ \), then \( a = 0 \) or \( b = 0 \).

We start our investigation of symmetric bi-derivations with the following results.

**Theorem 2.3.** Let \( M \) be a 2-torsion free prime Γ-ring satisfying the condition (*). Let \( D: M \times M \to M \) and \( d \) be a symmetric bi-derivation and the trace of \( D \), respectively. Suppose that \( d \) is commuting on \( M \), then \( M \) is commutative or \( D = 0 \).

**Proof.** We have
\[ [d(x), x]_α = 0, \text{ for all } x \in M, α \in Γ. \] (1)
The linearization of (1) gives us \( [d(x) + d(y) + 2D(x, y), x + y]_α = 0 \), which leads to
\[ [d(x), y]_α + [d(y), x]_α + 2[D(x, y), x]_α + 2[D(x, y), y]_α = 0 \text{ for all } x, y \in M, α \in Γ. \] (2)
Substituting $-x$ for $x$ in the relation above, we arrive at

$$[d(x), y]_\alpha - [d(y), x]_\alpha + 2[D(x, y), x]_\alpha - 2[D(x, y), y]_\alpha = 0 \text{ for all } x, y \in M, \alpha \in \Gamma. \quad (3)$$

From (2) and (3) we obtain

$$[d(x), y]_\alpha + 2[D(x, y), x]_\alpha = 0 \text{ for all } x, y \in M, \alpha \in \Gamma. \quad (4)$$

Replacing $y$ in (4) by $x\beta y$. Then by using the condition (*),

$$0 = [d(x), x\beta y]_\alpha + 2[d(x)\beta y + x\beta D(x, y), x]_\alpha$$

which, according to (4), implies

$$d(x)\beta [x, y]_\alpha = 0 \text{ for all } x, y \in M, \alpha, \beta \in \Gamma. \quad (5)$$

From the relation (5) and Lemma 2.1 one can conclude that $d(x) = 0$ or $[x, y]_\alpha = 0$ for all $x, y \in M, \alpha \in \Gamma$. If $[x, y]_\alpha = 0$, then $M$ is commutative. On the other hand, for any $x \notin Z(M)$, we have $[x, y]_\alpha \neq 0$. Therefore $d(x) = 0$ (note that for any fixed $x \in M, \alpha \in \Gamma$, a mapping $y \mapsto [x, y]_\alpha$ is a derivation). Let $x \in Z(M), y \notin Z(M)$. Then $x + y \notin Z(M)$ and $x - y \notin Z(M)$. Thus $0 = d(x + y) = d(x) + 2D(x, y)$ and $0 = d(x) - 2D(x, y)$. From these two relations, we have $4D(x, y) = 0$. By the 2-torsion freeness of $M$, we have

$$D(x, y) = 0 \text{ for all } x, y \in M.$$

The proof of the theorem is complete.

**Theorem 2.4.** Let $M$ be a 2 and 3-torsion free prime $\Gamma$-ring satisfying the condition (*). Let $D: M \times M \to M$ and $d$ be a symmetric bi-derivation and the trace of $D$, respectively. Suppose that $d$ is centralizing on $M$, then $M$ is commutative or $D = 0$.

**Proof** We have

$$[d(x), x]_\alpha \in Z(M) \text{ for all } x \in M, \alpha \in \Gamma. \quad (6)$$

By linearization we obtain

$$[d(x) + d(y) + 2D(x, y), x + y]_\alpha \in Z(M)$$

$$\Rightarrow [d(y), x]_\alpha + [d(x), y]_\alpha + 2[D(x, y), x]_\alpha \in Z(M) \text{ for all } x, y \in M, \alpha \in \Gamma. \quad (7)$$

since (6) holds. Replacing $x$ in the relation (7) by $-x$, we obtain

$$-d(y), x]_\alpha + [d(x), y]_\alpha - 2[D(x, y), y]_\alpha + 2[D(x, y), x]_\alpha \in Z(M) \text{ for all } x, y \in M, \alpha \in \Gamma. \quad (8)$$

Now (7) and (8) give us

$$[d(x), y]_\alpha + 2[D(x, y), x]_\alpha \in Z(M) \text{ for all } x, y \in M, \alpha \in \Gamma. \quad (9)$$

Replacing $y$ in (9) by $x\beta x$, we get

$$[d(x), x\beta x]_\alpha + 2[d(x)\beta x + x\beta d(x), x]_\alpha \in Z(M)$$

$$\Rightarrow [d(x), x]_\alpha \beta x + x\beta [d(x), x]_\alpha + 2[d(x), x]_\alpha \beta x + 2x\beta [d(x), x]_\alpha \in Z(M)$$

$$\Rightarrow 6[d(x), x]_\alpha \beta x \in Z(M) \text{ for all } x, y \in M, \alpha, \beta \in \Gamma. \quad (10)$$
Using (10), (6) and the assumptions that $M$ is 2 and 3-torsion free, we obtain

$$[d(x), x]_{\alpha} \beta [x, y]_{\alpha} = 0 \text{ for all } x, y \in M, \alpha, \beta \in \Gamma.$$  

Now, the relation above makes it possible to conclude, using the same arguments as in the proof of Theorem 2.3, that for any $x \notin Z(M)$ we have $[d(x), x]_{\alpha} = 0$.

In view of Theorem 2.3, the proof is complete.

**Theorem 2.5.** Let $M$ be a 2-torsion free prime $\Gamma$-ring satisfying the condition (*). Suppose there exist symmetric bi-derivations $D_1: M \times M \to M$ and $D_2: M \times M \to M$, such that $D_1(d_2(x), x) = 0$ holds for all $x \in M$, where $d_2$ denotes the trace of $D_2$.

Then $D_1 = 0$ or $D_2 = 0$.

**Proof.** By linearization of the relation

$$D_1(d_2(x), x) = 0 \text{ for all } x \in M. \tag{11}$$

we obtain according to (11),

$$D_1(d_2(y) + 2D_2(x, y), x + y) = 0$$

$$\Rightarrow D_1(d_2(y), x) + 2D_1(D_2(x, y), x) + D_1(d_2(x), y) + 2D_1(D_2(x, y), y) = 0 \text{ for all } x, y \in M.$$  

Replacing $x$ by $-x$ and comparing this new equation with the preceding equation we get

$$D_1(d_2(x), y) + 2D_1(D_2(x, y), x) = 0 \text{ for all } x, y \in M. \tag{12}$$

Let us replace $y$ by $xay$ in (12). Then

$$0 = D_1(d_2(x, xay) + 2D_1(D_2(x, xay), x)$$

$$= D_1(d_2(x), x)\alpha y + 2D_1(D_2(x, y), x) + 2d_2(x)\alpha y + d_2(x)\alpha D_2(x, y), x)$$

$$= x\alpha D_1(d_2(x), y) + 2D_1(D_2(x, y), x) + 2d_2(x)\alpha D_2(x, y) + 2d_2(x)\alpha D_2(x, y), x)$$

$$= x\alpha D_1(d_2(x), y) + 2d_2(x)\alpha D_2(x, y, x) + 2d_2(x)\alpha D_2(x, y) + 2d_2(x)\alpha D_2(x, y)$$

$$= 2d_2(x)\alpha D_1(x, y) + 2d_2(x)\alpha D_2(x, y).$$

In the above calculation we used (11) and (12). Thus we have

$$d_2(x)\alpha D_1(x, y) + d_1(x)\alpha D_2(x, y) = 0 \text{ for all } x, y \in M, \alpha \in \Gamma. \tag{13}$$

Let us replace $y$ in (13) by $y\beta x$. We get

$$0 = d_2(x)\alpha D_1(y\beta x, x) + d_1(x)\alpha D_2(y\beta x, x)$$

$$= d_2(x)\alpha D_1(y, x)\beta x + y\beta D_1(x) + d_1(x)\alpha D_2(x, y)\beta x + y\beta D_2(x)$$

$$= (d_2(x)\alpha D_1(x, y) + d_1(x)\alpha D_2(x, y))\beta x + d_1(x)\alpha y\beta D_2(x) + d_2(x)\alpha y\beta D_2(x)$$

$$= d_1(x)\alpha y\beta D_2(x) + d_2(x)\alpha y\beta D_2(x).$$

Thus, we have

$$d_1(x)\alpha y\beta D_2(x) + d_2(x)\alpha y\beta D_2(x) = 0 \text{ for all } x, y \in M, \alpha, \beta \in \Gamma. \tag{14}$$
Let us assume that $d_1$ and $d_2$ are both different from zero. In other words there exist elements $x_1, x_2 \in M$ such that $d_1(x_1) \neq 0$ and $d_2(x_2) \neq 0$. From (14) and Lemma 2.2, it follows that $d_1(x_2) = d_2(x_2) = 0$. Since $d_1(x_2) = 0$, the relation (13) reduces to $d_2(x_2) \alpha D_1(x_2, y) = 0$. Using this relation and Lemma 2.1, we obtain that $D_1(x_2, y) = 0$ holds for all $y \in M$ since $d_2(x_2) \neq 0$ (recall that a mapping $y \to D_1(x_2, y)$ is a derivation). In particular we have $D_1(x_2, x_1) = 0$. Similarly, we obtain $D_2(x_1, x_2) = 0$ holds as well. Let us write $y$ for $x_1 + x_2$. Then $d_1(y) = d_i(x_1 + x_2) = d_i(x_1) + d_i(x_2) + 2D_i(x_1, x_2) = d_i(x_1) \neq 0$. Similarly, we obtain $d_2(y) \neq 0$. But $d_i(y)$ and $d_2(y)$ cannot be both different from zero according to (14) and Lemma 2.2. Therefore we have proved that $d_1 = 0$ or $d_2 = 0$ which is the assertion of the theorem.

In case $D_1 = D_2$ Theorem 2.5 can be proved for semiprime $\Gamma$-rings.

**Theorem 2.6.** Let $M$ be a 2-torsion free semiprime $\Gamma$-ring. Suppose there exists such a symmetric bi-derivation $D: M \times M \to M$ that $D(d(x), x) = 0$ holds for all $x \in M$, where $d$ denotes the trace of $D$. Then $D = 0$.

**Proof.** In this case (14) reduces to $d(x) \alpha y \beta d(x) = 0$ for $x, y \in M$, $\alpha, \beta \in \Gamma$, which implies that $d(x) = 0$ for all $x \in M$, by semiprimeness of Posner [10] has proved a result which states that in case $M$ is a 2-torsion free prime $\Gamma$-ring and $D_1, D_2$ are nonzero derivations on $M$, then the mapping $x \to D_1(D_2(x))$ cannot be a derivation.

The result below was motivated by Posner’s result mentioned above.

**Theorem 2.7.** Let $M$ be a 2 and 3-torsion free prime $\Gamma$-ring satisfying the condition (*). Let $D_1: M \times M \to M$ and $D_2: M \times M \to M$ be symmetric bi-derivations. Suppose further that there exists a symmetric bi-additive mapping $B: M \times M \to M$ such that $d_i(d_j(x)) = f(x)$ holds for all $x \in M$, where $d_1$ and $d_2$ are the traces of $D_1$ and $D_2$, respectively, and $f$ is the trace of $B$. Then $D_1 = 0$ or $D_2 = 0$.

**Proof.** The linearization of the relation

$$d_1(d_2(x)) = f(x) \quad \text{for all } x \in M. \quad (15)$$

gives us

$$d_1(d_2(x) + d_2(y) + 2D_1(x, y)) = f(x) + f(y) + 2B(x, y)$$

and

$$d_1(d_2(x)) + d_1(d_2(y)) + 4d_1(D_2(x, y)) + 2D_1(d_2(x), d_2(y)) + 4D_1(d_2(x), D_2(x, y)) + 4D_1(d_2(y), D_2(x, y)) = f(x) + f(y) + 2B(x, y).$$

Using (15) we arrive at

$$2d_1(D_2(x, y)) + D_1(d_1(x), d_1(y)) + 2D_1(d_2(x), D_2(x, y)) + 2D_1(d_2(y), D_2(x, y)) = B(x, y).$$

Substituting in the equation above $x$ by $-x$ we obtain by comparing this new equation with the equation above that

$$2D_1(d_2(x), D_2(x, y)) + 2D_1(d_2(y), D_2(x, y)) = B(x, y) \quad \text{for all } x, y \in M. \quad (16)$$

Let us replace in (16) $x$ by $2x$. We have

$$8D_1(d_2(x), D_2(x, y)) + 2D_1(d_2(y), D_2(x, y)) = B(x, y) \quad \text{for all } x, y \in M. \quad (17)$$
By comparing (16) and (17) we obtain
\[ 6D_1(d_3(x), D_2(x, y)) = 0 \]
\[ \Rightarrow D_1(d_3(x), D_2(x, y)) = 0 \text{ for all } x, y \in M. \] (18)
since \( M \) is 2 and 3-torsion free. From (18) it follows that both terms on the left side of the relation (16) are zero, which means that \( B = 0 \). Hence (15) reduces to
\[ d_1(d_2(x)) = 0 \text{ for all } x \in M. \] (19)

Let in (18) \( y \) be \( y \alpha x \). We have
\[ 0 = D_1(d_2(x), D_2(x, y \alpha x)) \]
\[ = D_1(d_2(x), D_2(x, y \alpha x) + y \alpha d_2(x)) \]
\[ = D_1(d_2(x), D_2(x, y \alpha x) + D_1(d_2(x), y \alpha d_2(x))) \]
\[ = D_1(d_2(x), D_2(x, y)\alpha x + D_2(x, y)\alpha D_1(d_2(x), x) + D_1(d_2(x), y)\alpha d_2(x) + y \alpha d_1(d_2(x)) \]
for all \( x, y \in M, \alpha \in \Gamma \).

This implies
\[ D_1(d_2(x), y)\alpha d_2(x) + D_2(x, y)\alpha D_1(d_2(x), x) = 0 \text{ for all } x, y \in M, \alpha \in \Gamma. \] (20)
according to (18) and (19). Let us replace in (20) \( y \) by \( x \beta \). We have
\[ 0 = D_1(d_2(x), x \beta y)\alpha d_2(x) + D_2(x, x \beta y)D_1(d_2(x), x) \]
\[ = D_1(d_2(x), x)\alpha y \beta d_2(x) + x \beta D_1(d_2(x), y)\alpha d_2(x) + d_2(x)\alpha y \beta D_1(d_2(x), x) \]
\[ + x \alpha D_2(x, y)\beta D_1(d_2(x), x) \]
\[ = D_1(d_2(x), x)\alpha y \beta d_2(x) + d_2(x)\alpha y \beta D_1(d_2(x), x) + x \beta (D_1(d_2(x), y)\alpha d_2(x) \]
\[ + D_2(x, y)\alpha D_1(d_2(x), x)) \text{ for all } x, y \in M, \alpha, \beta \in \Gamma. \]

Now, by (20), we arrive finally at
\[ D_1(d_2(x), x)\alpha y \beta d_2(x) + d_2(x)\alpha y \beta D_1(d_2(x), x) = 0 \text{ for all } x, y \in M, \alpha, \beta \in \Gamma. \] (21)

From the relation above one can conclude that \( D_1(d_2(x), x) = 0 \) is fulfilled for all \( x \in M \).
Namely, if \( D_1(d_2(x), x) \neq 0 \) for some \( x \in M \), then \( d_2(x) = 0 \) according to (21) and Lemma 2.2, contrary to the assumption \( D_1(d_2(x), x) \neq 0 \). Therefore, since \( D_1(d_2(x), x) = 0 \) for all \( x \in M \), the proof of the theorem is complete since all the requirements of Theorem 2.5 are fulfilled.

In case \( D_1 = D_2 \) Theorem 2.7 can be proved for semi-prime \( \Gamma \)-rings.

**Theorem 2.8.** Let \( M \) be a 2, 3-torsion free semiprime \( \Gamma \)-ring satisfying the condition (*).
Let \( D: M \times M \to M \) and \( B: M \times M \to M \) be a symmetric bi-derivation and a symmetric bi-additive mapping, respectively. Suppose that \( d(d(x)) = f(x) \) holds for all \( x \in M \), where \( d \) is the trace of \( D \) and \( f \) is the trace of \( B \). Then \( D = 0 \).
Proof. Obviously, we can use the beginning of the proof of Theorem 2.5. In this case relations (18) and (19) can be written in the form

\[ D(d(x), D(x, y)) = 0 \text{ for all } x, y \in M. \]  

(22)

and

\[ d(d(x)) = 0 \text{ for all } x \in M. \]  

(23)

Let us write in (22) \( y \alpha z \) instead of \( y \). We have

\[ 0 = D(d(x), D(x, y \alpha z)) = D(d(x), D(x, y) \alpha z + y \alpha D(x, z)) \]

\[ = D(d(x), D(x, y) \alpha z) + D(d(x), y \alpha D(x, z)) \]

\[ = D(d(x), D(x, y) \alpha z) + D(d(x), y) \alpha D(x, z) + D(d(x), y \alpha D(x, z) + y \alpha D(d(x), D(x, z))) \text{ for all } x, y, z \in M, \alpha \in \Gamma. \]

Hence by (22) we have

\[ D(x, y) \alpha D(d(x), z) + D(d(x), y) \alpha D(x, z) = 0 \]

and, in particular, for \( z = d(x) \) we obtain

\[ D(d(x), y) \alpha D(x, d(x)) = 0 \text{ for all } x, y \in M, \alpha \in \Gamma. \]  

(24)

according to (23). Replace in (24) \( y \) by \( x \beta y \). We have

\[ 0 = D(d(x), x \beta y) \alpha D(x, d(x)) = D(d(x), x \beta y) \alpha D(x, d(x)) + x \beta D(d(x), y) \alpha D(x, d(x)) \]

which leads to

\[ D(d(x), x) \alpha y \beta D(d(x), x) = 0; \text{ for all } x \in M, \alpha, \beta \in \Gamma; \]

and we obtain \( D(d(x), x) = 0 \) for all \( x \in M \) by the semiprimeness of \( M \). Thus by Theorem 2.6 the proof is complete.

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