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# JORDAN DERIVATIONS ON COMPLETELY SEMIPRIME GAMMA–RINGS

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**ABSTRACT**

In this paper we prove that under a suitable condition every Jordan derivation on a 2-torsion free completely semiprime Γ-ring is a derivation.

**Keywords:** Derivation, Jordan derivation, Γ-ring, completely semiprime Γ-ring

## 1. Introduction

The concepts of derivation and Jordan derivation of a Γ-ring have been introduced by M. Sapanci and A. Nakajima in [8]. For the classical ring theories, Herstien [6], proved a well known result that every Jordan derivation in a 2-torsion free prime ring is a derivation. Bresar[2] proved this result in semiprime rings. In [8], Sapanci and Nakajima proved the same result in completely prime Γ-rings. C. Haetinger [4] worked on higher derivations on prime rings and extended this result to Lie ideals in a prime ring.

In this article, we have shown that every Jordan derivation of a 2-torsion free completely semiprime Γ-ring with the condition $a\alpha b\beta c=a\beta b\alpha c$, $\forall\ a, b, c \in$ M and $\alpha, \beta \in \Gamma$, is a derivation of M.

Let M and Γ be additive abelian groups. If there is a mapping $M \times \Gamma \times M \to M$ sending $(x, \alpha, y)$ into $x\alpha y$ such that the conditions

(i) $(x + y)\alpha z = x\alpha z + y\alpha z$, $x(\alpha + \beta) y = x\alpha y + x\beta y$, $x\alpha(y + z) = x\alpha y + x\alpha z$ and

(ii) $(x\alpha y)\beta z = x\alpha(y\beta z)$

are satisfied $\forall\ x, y, z \in$ M and $\alpha, \beta \in \Gamma$, then M is called a Γ-ring. This definition is due to Barnes [1]. A Γ-ring M is 2-torsion free if $2a = 0$ ($a \in$ M) implies $a = 0$. Besides M is called a semiprime Γ-ring if $a\Gamma M\Gamma a = 0$ (with $a \in$ M) implies $a = 0$. And, M is called completely semiprime if $a\Gamma a = 0$ ($a \in$ M) implies $a = 0$. Note that every completely semiprime Γ-ring is clearly a semiprime Γ-ring. We define $[a, b]_\alpha$ by $a\alpha b - b\alpha a$ which is known as a commutator of $a$ and $b$ with respect to $\alpha$. Let M be a Γ-ring. An additive mapping $d : M \to M$ is called a derivation if $d(a\alpha b) = d(a)\alpha b + a\alpha d(b)$, $\forall\ a, b \in$ M and $\alpha \in \Gamma$. And $d: M \to M$ is called a Jordan derivation if $d(a\alpha a) = d(a)\alpha a + a\alpha d(a)$, $\forall\ a \in$ M and $\alpha \in \Gamma$. Throughout the article, we use the condition $a\alpha b\beta c = a\beta b\alpha c$, ($a, b, c \in$ M and $\alpha, \beta \in \Gamma$) and refer to this condition as (*).

## 2. Some Consequences of Jordan Derivations on Completely Semiprime Γ-Ring

In this section, we develop some useful consequences regarding the Jordan derivation of a 2-torsion free completely semiprime Γ-ring which are very much needed for proving the main result.

**Lemma 2.1** Let M be a Γ-ring and let d be a Jordan derivation of M. Then $\forall a, b, c \in M$ and $\alpha, \beta \in \Gamma$, the following statements hold:

(i) $d(a\alpha b + b\alpha a) = d(a)\alpha b + d(b)\alpha a + a\alpha d(b) + b\alpha d(a)$.

(ii) $d(a\alpha b\beta a + a\beta b\alpha a) = d(a)\alpha b\beta a + d(a)\beta b\alpha a + a\alpha d(b)\beta a + a\beta d(b)\alpha a + a\alpha b\beta d(a) + a\beta b\alpha d(a)$.

In particular, if M is 2-torsion free and M satisfies the condition (*), then

(iii) $d(a\alpha b\beta a) = d(a)\alpha b\beta a + a\alpha d(b)\beta a + a\alpha b\beta d(a)$.

(iv) $d(a\alpha b\beta c + c\alpha b\beta a) = d(a)\alpha b\beta c + d(c)\alpha b\beta a + a\alpha d(b)\beta c + c\alpha d(b)\beta a + a\alpha b\beta d(c) + c\alpha b\beta d(a)$.

**Definition 1**. Let $d$ be a Jordan derivation of a Γ-ring M. Then $\forall a, b \in M$ and $\alpha \in \Gamma$, we define $G_\alpha(a, b) = d(a\alpha b) - d(a)\alpha b - a\alpha d(b)$. Thus we have $G_\alpha(b, a) = d(b\alpha a) - d(b)\alpha a - b\alpha d(a)$.

**Lemma 2.2** Let $d$ be a Jordan derivation of a Γ-ring M. Then $\forall a, b, c \in M$ and $\alpha, \beta \in \Gamma$, the following statements hold:

(i) $G_\alpha(b, a) = -G_\alpha(a, b)$ ; (*ii*) $G_\alpha(a + b, c) = G_\alpha(a, c) + G_\alpha(b, c)$;

(ii) $G_\alpha(a, b + c) = G_\alpha(a, b) + G_\alpha(a, c)$; (*iv*) $G_{\alpha+\beta}(a, b) = G_\alpha(a, b) + G_\beta(a, b)$.

*Remark*. $d$ is a derivation of a Γ-ring M if and only if $G_\alpha(a, b) = 0, \forall a, b \in M$ and $\alpha \in \Gamma$.

**Lemma 2.3** Let M be a 2-torsion free Γ-ring satisfying the condition (*) and let d be a Jordan derivation of *M*. Then

$$G_\alpha(a, b)\beta [a, b]_\alpha + [a, b]_\alpha \beta G_\alpha(a, b) = 0, \forall a, b \in M \text{ and } \alpha, \beta \in \Gamma.$$

Proof. (i) For any $a, b \in M$ and $\alpha, \beta \in \Gamma$, we have by using Lemma 2 .1(i)

$$\begin{aligned} W &= d(a\alpha b\beta b\alpha a + b\alpha a\beta a\alpha b) \\ &= d((a\alpha b)\beta(b\alpha a) + (b\alpha a)\beta(a\alpha b)) \\ &= d(a\alpha b)\beta b\alpha a + a\alpha b\beta d(b\alpha a) + d(b\alpha a)\beta a\alpha b + b\alpha a\alpha d(a\alpha b) \end{aligned}$$

On the other hand by using Lemma 2 .1(iii)

$$\begin{aligned} W &= d(a\alpha(b\beta b)\alpha a + b\alpha(a\beta a)\alpha b) \\ &= d(a\alpha(b\beta b)\alpha a) + d(b\alpha(a\beta a)\alpha b) \\ &= d(a)\alpha b\beta b\alpha a + a\alpha d(b\beta b)\alpha a + a\alpha b\beta b\alpha d(a) + d(b)\alpha a\beta a\alpha b + b\alpha d(a\beta a)\alpha b + b\alpha a\beta a\alpha d(b) \\ &= d(a)\alpha b\beta b\alpha a + a\alpha d(b)\beta b\alpha a + a\alpha b\beta d(b)\gamma b\alpha a + a\alpha b\beta d(b)\alpha a + a\alpha b\beta b\alpha d(a) + \\ &\quad d(b)\alpha a\beta a\alpha b + b\alpha d(a)\beta a\alpha b + b\alpha a\beta d(b)\gamma a\alpha b + b\alpha a\beta d(a)\alpha b + b\alpha a\beta a\alpha d(b) \end{aligned}$$

Equating two expressions for W and canceling the like terms from both sides, we get

$d(a\alpha b)\beta b\alpha a + a\alpha b\beta d(b\alpha a) + d(b\alpha a)\beta a\alpha b + b\alpha a\beta d(a\alpha b)$

$= d(a)\alpha b\beta b\alpha a + a\alpha d(b)\beta b\alpha a + a\alpha b\beta d(b)\alpha a + a\alpha b\beta b\alpha d(a) + d(b)\alpha a\beta a\alpha b +$

$b\alpha d(a)\beta a\alpha b + b\alpha a\beta d(a)\alpha b + b\alpha a\beta a\alpha d(b)$

This gives $d(a\alpha b)\beta b\alpha a - d(a)\alpha b\beta b\alpha a - a\alpha d(b)\beta b\alpha a + d(b\alpha a)\beta a\alpha b - d(b)\alpha a\beta a\alpha b - b\beta d(a)\ \beta a\alpha b + a\alpha b\beta d(b\alpha a) - a\alpha b\beta d(b)\alpha a - a\alpha b\beta b\alpha d(a) + b\alpha a\alpha d(a\alpha b) - b\alpha a\beta d(a)\alpha b - b\alpha a\beta a\alpha d(b) = 0$

This implies that $(d(a\alpha b) - d(a)\alpha b - a\alpha d(b))\alpha b\alpha a + (d(b\alpha a) - d(b)\alpha a - b\alpha d(a))\alpha a\alpha b + a\alpha b\alpha(d(b\alpha a) - d(b)\alpha a - b\alpha d(a)) + b\alpha a\alpha(d(a\alpha b) - d(a)\alpha b - a\alpha d(b)) = 0$

Now using the Definition 1, we obtain

$G_\alpha(a, b)\beta b\alpha a + G_\alpha(b, a)\beta a\alpha b + a\alpha b\beta G_\alpha(b, a) + b\alpha a\beta G_\alpha(a, b) = 0$

This implies that $G_\alpha(a, b)\beta[a, b]_\alpha + [a, b]_\alpha\beta G_\alpha(a, b) = 0, \forall a, b \in M, \alpha, \beta \in \Gamma.$

**Lemma 2.4** Let M be a 2-torsion free completely semiprime Γ-ring and let $a, b \in$ M, $\alpha \in \Gamma$.

If $a\alpha b + b\alpha a = 0$, then $a\alpha b = 0 = b\alpha a$.

**Proof.** Let $\delta \in \Gamma$ be any element.

Using the relation $a\alpha b = -b\alpha a$ repeatedly, we get

$(a\alpha b)\delta(a\alpha b) = -(b\alpha a)\delta(a\alpha b) = -(b\alpha(a\delta)a)\alpha b = (a(\alpha a\delta)b)\alpha b$

$= a\alpha(a\delta b)\alpha b = -a\alpha(b\delta a)\alpha b = -(a\alpha b)\delta(a\alpha b)$

This implies, $2((a\alpha b)\delta(a\alpha b)) = 0$.

Since M is 2-torsion free, $(a\alpha b)\delta(a\alpha b) = 0$

Therefore, $(a\alpha b)\Gamma(a\alpha b) = 0$

By the completely semiprimeness of M, we get $a\alpha b = 0$

Similarly, it can be shown that $b\alpha a = 0$.

**Corollary 2.1** Let M be a 2-torsion free completely semiprime Γ-ring satisfying the condition (*) and let $d$ be a Jordan derivation of M. Then $\forall a, b \in$ M and $\alpha, \beta \in \Gamma$ :

(i) $G_\alpha(a, b)\beta[a, b]_\alpha = 0$; (ii)$[a, b]_\alpha \beta G_\alpha(a, b) = 0$

**Proof**. Applying the result of Lemma 2 .4 in that of Lemma 2 .3, we obtain these results.

**Lemma 2.5** Let M be a 2-torsion free completely semiprime Γ-ring satisfying the condition (*) and let $d$ be a Jordan derivation of M. Then $\forall\ a, b, x, y \in$ M and $\alpha, \beta, \gamma \in \Gamma$ :

(i) $G_\alpha(a, b)\beta[x, y]_\alpha = 0$; (ii)$[x, y]_\alpha \beta G_\alpha (a, b) = 0$

(iii) $G_\alpha(a, b)\beta[x, y]_\gamma = 0$; (iv)$[x, y]_\gamma \beta G_\alpha(a, b) = 0$

**Proof.** (i) If we substitute $a + x$ for $a$ in the Corollary 2.1 (v), we get $G_\alpha(a + x, b)\beta[a + x, b]_\alpha = 0$

Thus $G_\alpha(a, b)\beta[a, b]_\alpha+G_\alpha(a, b)\beta[x, b]_\alpha+G_\alpha(x, b)\beta[a, b]_\alpha+G_\alpha(x, b)\beta[x, b]_\alpha= 0$

By using Corollary 2 .1 (v), we have $G_\alpha(a, b)\beta[x, b]_\alpha + G_\alpha(x, b)\beta[a, b]_\alpha = 0$

Thus, we obtain

$(G_\alpha(a, b)\beta[x, b]_\alpha)\beta(G_\alpha(a, b)\beta[x, b]_\alpha) = -G_\alpha(a, b)\beta[x, b]_\alpha \beta G_\alpha(x, b)\beta[a, b]_\alpha = 0$

Hence, by the completely semiprimeness of M, we get $G_\alpha(a, b)\beta[x, b]_\alpha = 0$

Similarly, by replacing $b + y$ for $b$ in this result, we get $G_\alpha(a, b)\beta[x, y]_\alpha = 0$

(ii) Proceeding in the same way as described above by the similar replacements successively

in Corollary 2.1(vi), we obtain $[x, y]_\gamma \beta G_\alpha(a, b) = 0,\ \forall a, b, x, y \in M, \alpha, \beta \in \Gamma$

(iii) Replacing $\alpha + \gamma$ for $\alpha$ in (i), we get $G_{\alpha+\gamma}(a, b)\beta[x, y]_{\alpha+\gamma} = 0$

This implies $(G_\alpha(a, b) + G_\gamma(a, b)_\alpha)\beta([x, y]_\alpha + [x, y]_\gamma) = 0$

Therefore $G_\alpha(a, b)\beta[x, y]_\alpha\ G_\alpha(a, b)\beta[x, y]_\gamma + G_\gamma(a, b)\beta[x, y]_\alpha + G_\gamma(a, b)\beta[x, y]_\gamma = 0$

Thus by using Corollary 2.1 (vi), we get $G_\alpha(a, b)\beta[x, y]_\gamma + G_\gamma(a, b)\beta[x, y]_\alpha = 0$

Thus, we obtain

$(G_\alpha(a, b)\beta[x, y]_\gamma)\beta(G_\alpha(a, b)\beta[x, y]_\gamma) = -G_\alpha(a, b)\beta[x, y]_\gamma \beta G_\gamma(a, b)\beta[x, y]_\alpha = 0$

Hence, by the completely semiprimeness of M, we obtain $G_\alpha(a, b)\beta[x, y]_\gamma = 0$

(iv) As in the proof of (iii), the similar replacement in (ii) produces (iv).

**Lemma 2.6** Every completely semiprime Γ-ring contains no nonzero nilpotent ideal.

**Corollary 2.2** Completely Semiprime Γ-ring has no nonzero nilpotent element.

**Lemma 2.7** The center of a completely semiprime Γ-ring does not contain any nonzero nilpotent element.

## 3. Jordan Derivations on Completely Semiprime Γ-ring

We are now ready to prove our main result as follows:

**Theorem 3.1** Let M be a 2-torsion free completely semiprime Γ-ring satisfying the condition (*) and let $d$ be a Jordan derivation of M. Then $d$ is a derivation of M.

**Proof.** Let $d$ be a Jordan derivation of a 2-torsion free completely semiprime Γ-ring M and let $a, b, y \in$ M and $\alpha, \beta, \gamma \in \Gamma$. Then by Lemma 2 .5(iii), we get

$$[G_\alpha(a, b), y]_\beta \gamma[G_\alpha(a, b), y]_\beta = (G_\alpha(a, b)\beta y - y\beta G_\alpha(a, b))\gamma[G_\alpha(a, b), y]_\beta$$

$= G_\alpha(a, b)\beta y\gamma[G_\alpha(a, b), y]_\beta - y\beta G_\alpha(a, b)\gamma[G_\alpha(a, b), y]_\beta = 0$

Since $\beta y\gamma \in$ M and $G_\alpha(a, b) \in M$, $\forall a, b, y \in$ M and $\alpha, \beta, \gamma \in \Gamma$.

By the completely semiprimeness of M, $[G_\alpha(a, b), y]_\beta = 0$, where $G_\alpha(a, b) \in M$, $\forall a, b, y \in M$ and $\alpha, \beta \in \Gamma$.

Therefore, $G_\alpha(a, b) \in Z(M)$, the center of M.

Now, by Lemma 2.5(iii), we have $G_\alpha(a, b)\gamma[x, y]_\beta = 0$ (1)

Also, by Lemma 2.5(iv), we have $[x, y]_\beta \gamma G_\alpha(a, b) = 0$ (2)

Thus, we have

$$
\begin{aligned}
2 G_\alpha(a, b)\gamma G_\alpha(a, b) &= G_\alpha(a, b)\gamma(G_\alpha(a, b + G_\alpha(a, b))\\
&= G_\alpha(a, b)\gamma(G_\alpha(a, b) - G_\alpha(b, a))\\
&= G_\alpha(a, b)\gamma(d(a\alpha b) - d(a)\alpha b - a\alpha d(b) - d(b\alpha a) + d(b)\alpha a + b\alpha d(a))\\
&= G_\alpha(a, b)\gamma(d(a\alpha b - b\alpha a) + (b\alpha d(a) - d(a)\alpha b) + (d(b)\alpha a - a\alpha d(b)))\\
&= G_\alpha(a, b)\gamma(d([a, b]_\alpha) + [b, d(a)]_\alpha + [d(b), a]_\alpha)\\
&= G_\alpha(a, b)\gamma d([a, b]_\alpha) - G_\alpha(a, b)\gamma[d(a), b]_\alpha - G_\alpha(a, b)\gamma[a, d(b)]_\alpha
\end{aligned}
$$

Since $d(a), d(b) \in M$, by using Lemma 2.5(i) and (1), we get

$$G_\alpha(a, b)\gamma[d(a), b]_\alpha = G_\alpha(a, b)\gamma[a, d(b)]_\alpha = 0$$

Thus $2G_\alpha(a, b)\gamma G_\alpha(a, b) = G_\alpha(a, b)\gamma\, d([a, b]_\alpha)$ (3)

Adding (1) and (2), we obtain $G_\alpha(a, b)\, \gamma[x, y]_\beta + [x, y]_\beta\, \gamma G_\alpha(a, b) = 0.$

Then by Lemma 2 .1(i) with the use of (1), we have

$$
\begin{aligned}
0 &= d(G_\alpha(a, b)\, \gamma[x, y]_\beta + [x, y]_\beta\, \gamma G_\alpha(a, b))\\
&= d(G_\alpha(a, b))\, \gamma[x, y]_\beta + d([x, y]_\beta)\, \gamma G_\alpha(a, b) + G_\alpha(a, b)\gamma d([x, y]_\beta) + [x, y]_\beta\, \gamma d(G_\alpha(a, b))\\
&= d(G_\alpha(a, b))\, \gamma[x, y]_\beta + 2\, G_\alpha(a, b)\gamma d([x, y]_\beta) + [x, y]_\beta\, \gamma d(G_\alpha(a, b))
\end{aligned}
$$

Since $G_\alpha(a, b) \in Z(M)$ and therefore $d([x, y]_\beta)G_\alpha(a, b) = G_\alpha(a, b)d([x, y]_\beta)$

Hence, we get

$2\, G_\alpha(a, b)\gamma d([x, y]_\beta) = -d(G_\alpha(a, b))\, \gamma[x, y]_\beta - [x, y]_\beta\gamma d(G_\alpha(a, b))$ (4)

Then from (3) and (4) we have

$$4G_\alpha(a, b)\gamma G_\alpha(a, b) = 2\, G_\alpha(a, b)\, \gamma d([a, b]_\alpha) = -d(G_\alpha(a, b))\gamma[a, b]_\alpha - [a, b]_\alpha\, \gamma d(G_\alpha(a, b))$$

Thus we obtain

$$
\begin{aligned}
4G_\alpha(a, b)\gamma G_\alpha(a, b)\gamma G_\alpha(a, b) = &-d(G_\alpha(a, b))\gamma[a, b]_\alpha\, \gamma G_\alpha(a, b)\\
&-[a, b]_\alpha\, \gamma d(G_\alpha(a, b))\gamma G_\alpha(a, b)
\end{aligned}
$$

Here, we have by Corollary 2 .1 (vi) $d(G_\alpha(a, b))\, \gamma[a, b]_\alpha\, \gamma G_\alpha(a, b) = 0$

and also, by Lemma 2.5(iv) $[a, b]_\alpha\, \gamma d(G_\alpha(a, b))\gamma G_\alpha(a, b) = 0$.

Since $d(G_\alpha(a, b)) \in M$, $\forall a, b \in M$ and $\alpha \in \Gamma$.

So, we get $4G_\alpha(a, b)\gamma G_\alpha(a, b)\gamma G_\alpha(a, b) = 0$.

Therefore, $4(G_\alpha(a, b)\gamma)^2 G_\alpha(a, b) = 0$.

Since M is 2-torsion free, so we have $(G_\alpha(a, b)\gamma)^2 G_\alpha(a, b) = 0$

But, it follows that $G_\alpha(a, b)$ is a nilpotent element of the Γ-ring M.

Since by Lemma 2.7, the center of a completely semiprime Γ-ring does not contain any nonzero nilpotent element, so we get $G_\alpha(a, b) = 0$, $\forall a, b \in$ M and $\alpha \in \Gamma$.

It means that, *d* is a derivation of M. Which is the required result.

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