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GENERALIZATION OF TITCHMARSH'S THEOREM FOR THE FOURIER TRANSFORM IN THE SPACE $L^2(\mathbb{R})$

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ABSTRACT

Using the Steklov function, we obtain a generalization of Titchmarsh's Theorem for the Fourier tranform for functions satisfying the Fourier-Lipschitz condition in the space $L^2(\mathbb{R})$.

Keywords: Fourier transform; Steklov function.

1. Introduction and preliminaries

The integral Fourier transform, as well as Fourier series, is widely used in various fields of calculus, computational mathematics, mathematical physics, etc. Certain applications of this transform are described in a number of fundamental monographs (e.g., see [3], [4], [6]).

Titchmarsh's ([7], Theorem 85) characterized the set of functions in $L^2(\mathbb{R})$ satisfying the Cauchy Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transform, namely we have the following.

Theorem 1.1 [7] Let $\alpha \in (0, 1)$ and assume that $f \in L^2(\mathbb{R})$. Then the following are equivalents:

1. $\| f(t+h) - f(t) \|_{L^2(\mathbb{R})} = O(h^{\alpha})$ as $h \to 0$

2. $\int_{|\lambda| \geq r} \hat{f}(\lambda)^2 d\lambda = O(r^{-2\alpha})$ as $r \to \infty$

where $\hat{f}$ stands for the Fourier transform of f.

The main aim of this paper is to establish a generalization of Theorem 1.1.

Assume that $L^p(\mathbb{R})$ $(p \geq 1)$ is the space of p-power integrable functions f with the norm

$$\| f \|_p = \left(\int_{-\infty}^{\infty} | f(x) |^p dx \right)^{1/p}.$$

It is well known that Fourier transform of a function $f \in L^1(\mathbb{R})$ is defined by the integral

$$\hat{f}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ixt} dx .$$

Thus, we define a linear operator $\hat{f}$ on the space $L^1(\mathbb{R})$ such that each function in this space is put in correspondence with its Fourier transform, which, generally speaking, does not belong to $L^1(\mathbb{R})$.

In 1910, Plancherel was the first to construct the Fourier operator $\hat{f}$ for the class $L^2(\mathbb{R})$. He proved the following remarkable theorem establishing the equivalence of the function $f \in L^2(\mathbb{R})$ and its Fourier transform (see [5]).

The inverse Fourier transform

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(t) e^{ixt} dt .$$

Parseval's equality

$$\int_{-\infty}^{\infty} \left| \hat{f}(t) \right|^2 dt = \int_{-\infty}^{\infty} \left| f(x) \right|^2 dx .$$

In $L^2(\mathbb{R})$, consider the operator (Steklov's function)

$$F_h f(x) = \frac{1}{2h} \int_{x-h}^{x+h} f(t) dt, \, h > 0$$

The finite differences of the first and higher orders are defined as follows.

$$\Delta_h f(x) = F_h f(x) - f(x) = (F_h - I) f(x),$$

$$\Delta_h^k f(x) = \Delta_h (\Delta_h^{k-1} f(x)) = (F_h - I)^k f(x) = \sum_{i=0}^{\infty} (-1)^{k-i} \binom{k}{i} F_h^i f(x),$$

where $F_h^0 f(x) = f(x), \; F_h^i f(x) = F_h(F_h^{i-1} f(x))$ (i = 1,2,...,k and k = 1,2,...), I is a unit operator in $L^2(\mathbb{R})$,

Consider the Sobolev space

$$W_2^k = \{ f \in L^2(\mathbb{R}); \; f^{(j)} \in L^2(\mathbb{R}), \; j = 1,2,\ldots,k \}$$

Since in [2], we have

$$F_h f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\sin(ht)}{ht} \hat{f}(t) e^{ixt} dt .$$

It follows from

$$F_h f(x) - f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \left(\frac{\sin(ht)}{ht} - 1 \right) \hat{f}(t) e^{ixt} dt,$$

and Parseval's equality that

$$\left\| F_h f(x) - f(x) \right\|_2^2 = \int_{-\infty}^{\infty} \left(1 - \frac{\sin(ht)}{ht} \right)^2 \left| \hat{f}(t) \right|^2 dt .$$

Hence, for any function $f \in W_2^k$, we have

$$\left\| \Delta_h^k f^{(r)}(x) \right\|_2^2 = \int_{-\infty}^{\infty} t^{2r} \left(1 - \frac{\sin(ht)}{ht} \right)^{2k} \left| \hat{f}(t) \right|^2 dt \quad . \tag{1}$$

For $p \geq -\dfrac{1}{2}$, we introduce the normalized Bessel function of the first kind j_p defined by

$$j_p(x) = \Gamma(p+1) \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{n! \Gamma(n+p+1)} \tag{2}$$

Moreover, from (2) we see that

$$\lim_{x \to 0} \frac{j_p(x) - 1}{x^2} = \frac{-\Gamma(p+1)}{4\Gamma(p+2)} \neq 0,$$

which implies that, there exist $c > 0$ and $\eta > 0$ satisfying

$$|x| \leq \eta \Rightarrow \left| j_p(x) - 1 \right| \geq c |x|^2 \tag{3}$$

Using the relation (2), we obtain

$$j_{1/2}(x) = \frac{\sin x}{x} \tag{4}$$

From [1], we have

$$\left| j_p(x) \right| \leq 1 \tag{5}$$

and

$$1 - j_p(x) = O(x^2), \ 0 \leq x \leq 1 \tag{6}$$

2. Main Results

In this section we give the main result of this paper. We need first to define the Fourier-Lipschitz class.

Definition 2.1 : Let $\alpha \in (0,k)$. A function $f \in W_2^k$ is said to be in the Fourier-Lipschitz class, denoted by Lip(α , 2), if

$$\left\|\Delta_h^k f^{(r)}(x)\right\|_2 = O(h^\alpha) \text{ as } h \to 0 .$$

Theorem 2.2 : Let $f \in W_2^k$. Then the following are equivalents

1. $f \in Lip(\alpha, 2)$
2. $\int_{|t|\geq N} t^{2r} \left|\hat{f}(t)\right|^2 dt = O(N^{-2\alpha})$ as $N \to +\infty$

Proof. (1) $\Rightarrow$ (2): Assume that $f \in Lip(\alpha, 2)$. Then we obtain

$$\left\|\Delta_h^k f^{(r)}(x)\right\|_2 = O(h^{2\alpha}) \text{ as } h \to 0 .$$

From formula (1), we have

$$\left\|\Delta_h^k f^{(r)}(x)\right\|_2^2 = \int_{-\infty}^{\infty} t^{2r} \left|1 - \frac{\sin(ht)}{ht}\right|^{2k} \left|\hat{f}(t)\right|^2 dt$$

By (4), we have

$$\left\|\Delta_h^k f^{(r)}(x)\right\|_2^2 = \int_{-\infty}^{\infty} t^{2r} \left|1 - j_{1/2}(ht)\right|^{2k} \left|\hat{f}(t)\right|^2 dt$$

Formula (3) gives

$$\int_{\frac{\eta}{2h}\leq|t|\leq\frac{\eta}{h}} t^{2r} \left|1 - j_{1/2}(ht)\right|^{2k} \left|\hat{f}(t)\right|^2 dt \geq \frac{c^{2k}\eta^{4k}}{2^{4k}} \int_{\frac{\eta}{2h}\leq|t|\leq\frac{\eta}{h}} t^{2r} \left|\hat{f}(t)\right|^2 dt .$$

There exists then a positive constant C

$$\int_{\frac{\eta}{2h}\leq|t|\leq\frac{\eta}{h}} t^{2r} \left|\hat{f}(t)\right|^2 dt \leq C \int_{-\infty}^{\infty} t^{2r} \left|1 - j_{1/2}(ht)\right|^{2k} \left|\hat{f}(t)\right|^2 dt$$

$$\leq Ch^{2\alpha}$$

Then

$$\int_{N\leq|t|\leq 2N} t^{2r} \left|\hat{f}(t)\right|^2 dt \leq CN^{-2\alpha}$$

Furthermore, we have

$$\int_{|t|\geq N} t^{2r} \left|\hat{f}(t)\right|^2 dt = \sum_{j=0}^{\infty} \int_{2^j N\leq|t|\leq 2^{j+1}N} t^{2r} \left|\hat{f}(t)\right|^2 dt$$

$$\leq C\sum_{j=0}^{\infty}(2^j N)^{-2\alpha}$$

$$\leq CN^{-2\alpha}$$

This proves that

$$\int_{|t|\geq N} t^{2r}\left|\hat{f}(t)\right|^2 dt = O(N^{-2\alpha}) \text{ as } N\to+\infty$$

(2) $\Rightarrow$ (1): Suppose now that

$$\int_{|t|\geq N} t^{2r}\left|\hat{f}(t)\right|^2 dt = O(N^{-2\alpha}) \text{ as } N\to+\infty .$$

We have to show that

$$\int_{-\infty}^{\infty} t^{2r}\left|1-j_{1/2}(ht)\right|^{2k}\left|\hat{f}(t)\right|^2 dt = (h^{2\alpha}) \text{ as } h\to 0 .$$

We write

$$\int_{-\infty}^{\infty} t^{2r}\left|1-j_{1/2}(ht)\right|^{2k}\left|\hat{f}(t)\right|^2 dt = \mathrm{I}_1+\mathrm{I}_2 ,$$

Where

$$\mathrm{I}_1 = \int_{|t|<1/h} t^{2r}\left|1-j_{1/2}(ht)\right|^{2k}\left|\hat{f}(t)\right|^2 dt$$

and

$$\mathrm{I}_2 = \int_{|t|\geq 1/h} t^{2r}\left|1-j_{1/2}(ht)\right|^{2k}\left|\hat{f}(t)\right|^2 dt$$

Firstly from (5) we see that

$$I_2 \leq 4^k \int_{|t|\geq 1/h} t^{2r}\left|\hat{f}(t)\right|^2 = O(h^{2\alpha})$$

Setting

$$\psi(s) = \int_s^{\infty} t^{2r}\left|\hat{f}(t)\right|^2 dt ,$$

and using integration by parts, we obtain

$$\int_0^s t^{2r}\left|\hat{f}(t)\right|^2 dt = \int_0^s -t^{2r}\psi'(t)dt$$

$$= -s^{2k}\psi(s) + 2k\int_0^s t^{2k-1}\psi(t)dt$$

$$\leq 2k\int_0^s O(t^{2k-1-2\alpha})dt$$

$$=O(s^{2k-2\alpha}).$$

We use formula (6), then

$$\int_{-\infty}^{\infty} t^{2r}\left|1-j_{1/2}(ht)\right|^{2k}\left|\hat{f}(t)\right|^{2}dt=O(h^{2k}\int_{|t|<1/h} t^{2k}t^{2r}\left|\hat{f}(t)\right|^{2}dt)+O(h^{2\alpha})$$

$$=O(h^{2k}h^{2\alpha-2k})+O(h^{2\alpha})$$

$$=O(h^{2\alpha})$$

and this ends the proof.

Corollary 2.3 : Let $f\in W_2^k$, and $f\in Lip(\alpha,2)$, Then

$$\int_{|t|\geq N}\left|\hat{f}(t)\right|^{2}dt=O(N^{-2\alpha-2r}).$$

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