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# ON DERIVATIONS IN PRIME GAMMA-NEAR-RINGS

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**ABSTRACT**

Let $N$ be a non zero-symmetric left $\Gamma$-near-ring. If $N$ is a prime $\Gamma$-near-ring with nonzero derivations $D_1$ and $D_2$ such that $D_1(x)\alpha D_2(y) = D_2(x)\alpha D_1(y)$ for every $x, y \in N$ and $\alpha \in \Gamma$, then we prove that $N$ is an abelian $\Gamma$-near-ring. Again if $N$ is a 2-torsion free prime $\Gamma$-near-ring and $D_1$ and $D_2$ are derivations satisfying $D_1(x)\alpha D_2(y) = D_2(x)\alpha D_1(y)$ for every $x, y \in N$ and $\alpha \in \Gamma$, then we prove that $D_1D_2$ is a derivation on $N$ if and only if $D_1 = 0$ or $D_2 = 0$.

**Key words**: Prime $\Gamma$-near-rings, semiprime $\Gamma$-near-rings, $N$-subsets, derivations.

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## 1. Introduction

In [2] Bell and Mason introduced the notion of derivations in near-rings. They obtained some basic properties of derivations in near-rings. Then Mustafa [11] investigated some commutativity conditions for a $\Gamma$-near-ring with derivations. Cho [5] studied some characterizations of $\Gamma$-near-rings and some regularity conditions. In classical ring theory, Posner [9], Herstein [6], Bergen [4], Bell and Daif [1] studied derivations in prime and semiprime rings and obtained some commutativity properties of prime rings with derivations. In near ring theory, Bell and Mason [2], and also Cho [10] worked on derivations in prime and semiprime near-rings.

In this paper, we slightly extend the results of Cho [10] in prime $\Gamma$-near-rings with certain conditions by using derivations.

A $\Gamma$-near-ring is a triple $(N, +, \Gamma)$ where

(i) $(N, +)$ is a group (not necessarily abelian),

(ii) $\Gamma$ is a non-empty set of binary operations on N such that for each $\alpha \in \Gamma$, $(N, +, \alpha)$ is a left near-ring.

(iii) $a\alpha(b\beta c) = (a\alpha b)\beta c$, for all $a, b, c \in N$ *and* $\alpha, \beta \in \Gamma$.

Exactly speaking, it is a *left* $\Gamma$-*near-ring* because it satisfies the left distributive law. We will use the word $\Gamma$-*near-ring* to mean *left* $\Gamma$-*near-ring.* For a near-ring $N$, the set $N_0 = \{a \in N\colon 0\alpha a = 0, \alpha \in \Gamma\}$ is called the *zero-symmetric part* of $N$. A $\Gamma$-near-ring $N$ is said to be *zero-symmetric* if $N = N_0$. Throughout this paper, $N$ will denote a zero-symmetric left

$\Gamma$-near-ring. A $\Gamma$-near-ring $N$ is called a prime $\Gamma$-near-ring if $N$ has the property that for $a, b \in N$, $a\Gamma N\Gamma b = \{0\}$ implies $a = 0$ or $b = 0$. $N$ is called a semiprime $\Gamma$-near-ring if $N$ has the property that for $a \in N$, $a\Gamma N\Gamma a = \{0\}$ implies $a = 0$. A nonempty subset $U$ of $N$ is called a right $N$-subset (resp. left $N$-subset) if $U\Gamma N \subset U$ (resp. $N\Gamma U \subset U$), and if $U$ is both a right $N$-subset and a left $N$-subset, it is said to be an $N$-subset of $N$. An ideal of $N$ is a subset $I$ of $N$ such that (i) $(I, +)$ is a normal subgroup of $(N, +)$, (ii) $a\Gamma(I + b) - a\Gamma b \subset I$ for all $a, b \in N$, (iii) $(I + a)\Gamma b - a\Gamma b \subset I$ for all $a, b \in N$. If $I$ satisfies (i) and (ii) then it is called a left ideal of $N$. If $I$ satisfies (i) and (iii) then it is called a right ideal of $N$.

On the other hand, a (two-sided) $N$-subgroup of $N$ is a subset $H$ of $N$ such that

(i) $(H, +)$ is a subgroup of $(N, +)$, (ii) $N\Gamma H \subset H$, and (iii) $H\Gamma N \subset H$. If H satisfies (i) and (ii) then it is called a left $N$-subgroup of $N$. If $H$ satisfies (i) and (iii) then it is called a right $N$-subgroup of $N$. Note that normal $N$-subgroups of $N$ are not equivalent to ideals of $N$. Every right ideal of $N$, right $N$-subgroup of $N$ and right semigroup ideal of $N$ are right $N$-subsets of $N$, and symmetrically, we can apply for the left case. A derivation $D$ on $N$ is an additive endomorphism of $N$ with the property that for all $a, b \in N$ and Ǚ∈Γ, *D*(*a*Ǚ*b*) = *a*Ǚ*D*(*b*) + *D*(*a*)Ǚ*b*.

## 2. Derivations in prime Γ-near-rings

A $\Gamma$-near-ring $N$ is called abelian if $(N, +)$ is abelian, and 2-torsion free if for all $a \in N$, $2a = 0$ implies $a = 0$.

**Lemma 2.1**. Let $D$ be an arbitrary additive endomorphism of $N$. Then *D*(*a*Ǚ*b*) = *a*Ǚ*D*(*b*) + *D*(*a*)Ǚ*b* if and only if *D*(*a*Ǚ*b*) = *D*(*a*)Ǚ*b* + *a*Ǚ*D*(*b*) for all $a, b \in N$ and Ǚ∈Γ.

**Proof**. Suppose that *D*(*a*Ǚ*b*) = *a*Ǚ*D*(*b*) + *D*(*a*)Ǚ*b*, for all $a, b \in N$ and Ǚ∈Γ. For Ǚ∈Γ and from

*a*Ǚ(*b* + *b*) = *a*Ǚ*b* + *a*Ǚ*b* and $N$ satisfies left distributive law

*D*(*a*Ǚ(*b* + *b*)) = *a*Ǚ*D*(*b* + *b*) + *D*(*a*)Ǚ(*b* + *b*) = *a*Ǚ(*D*(*b*) + *D*(*b*)) + *D*(*a*)Ǚ*b* + *D*(*a*)Ǚ*b*

= *a*Ǚ*D*(*b*) + *a*Ǚ*D*(*b*) + *D*(*a*)Ǚ*b* + *D*(*a*)Ǚ*b*

and

*D*(*a*Ǚ*b* + *a*Ǚ*b*) = *D*(*a*Ǚ*b*) + *D*(*a*Ǚ*b*) = *a*Ǚ*D*(*b*) + *D*(*a*)Ǚ*b* + *a*Ǚ*D*(*b*) + *D*(*a*)Ǚ*b*.

Comparing these two equalities, we have *a*Ǚ*D*(*b*) + *D*(*a*)Ǚ*b* = *D*(*a*)Ǚ*b* + *a*Ǚ*D*(*b*). Hence *D*(*a*Ǚ*b*) = *D*(*a*)Ǚ*b* + *a*Ǚ*D*(*b*), for $a, b \in N$, Ǚ∈Γ.

Conversely, suppose that *D*(*a*Ǚ*b*) = *D*(*a*)Ǚ*b* + *a*Ǚ*D*(*b*), for all $a, b \in N$ and Ǚ∈Γ. Then from *D*(*a*Ǚ(*b* + *b*)) = *D*(*a*Ǚ*b* + *a*Ǚ*b*) and the above calculation of this equality, we can induce that *D*(*a*Ǚ*b*) = *a*Ǚ*D*(*b*) + *D*(*a*)Ǚ*b*, for $a, b \in N$, Ǚ∈Γ.

**Lemma 2.2**. Let $D$ be a derivation on $N$. Then $N$ satisfies the following right distributive laws: for all $a, b, c \in N$ and Ǚ, б∈Γ,

{*a*Ǚ*D*(*b*) + *D*(*a*)Ǚ*b*}б*c* = *a*Ǚ*D*(*b*)б*c* + *D*(*a*)Ǚ*b*б*c*,

{*D*(*a*)Ǚ*b* + *a*Ǚ*D*(*b*)}б*c* = *D*(*a*)Ǚ*b*б*c* + *a*Ǚ*D*(*b*)б*c*,

**Proof.** From the calculation for $D((a\alpha b)\beta c) = D(a\alpha(b\beta c))$ for all $a, b, c \in N$ and $\alpha, \beta \in \Gamma$ and Lemma 2.1, we can induce our result.

**Lemma 2.3**. Let $N$ be a prime $\Gamma$-near-ring and let $U$ be a nonzero $N$-subset of $N$. If $a$ be an element of $N$ such that $U\Gamma a = \{0\}$ (or $a\Gamma U = \{0\}$), then $a = 0$.

**Proof**. Since $U \neq \{0\}$, there exist an element $u \in U$ such that $u \neq 0$. Consider that

$u\Gamma N\Gamma a \subset U\Gamma a = \{0\}$. Since $u \neq 0$ and $N$ is a prime $\Gamma$-near-ring, we have that $a = 0$.

**Corollary 2.4**. Let $N$ be a semiprime $\Gamma$-near-ring and let $U$ be a nonzero $N$-subset of $N$. If $a$ be an element of $N(U)$ such that $U\Gamma a\Gamma a = \{0\}$ (or $a\Gamma a\Gamma U = \{0\}$), where $N(U)$ is the normalizer of $U$, then $a = 0$.

**Lemma 2.5**. Let $N$ be a prime $\Gamma$-near-ring and $U$ a nonzero $N$-subset of $N$. If $D$ is a nonzero derivation on $N$. Then (i) If $a, b \in N$ and $a\Gamma U\Gamma b = \{0\}$, then $a = 0$ or $b = 0$.

(ii) If $a \in N$ and $D(U)\Gamma a = \{0\}$, then $a = 0$. (iii) If $a \in N$ and $a\Gamma D(U) = \{0\}$, then $a = 0$.

**Proof**. (i) Let $a, b \in N$ and $a\Gamma U\Gamma b = \{0\}$. Then $a\Gamma U\Gamma N\Gamma b \subset a\Gamma U\Gamma b = \{0\}$. Since $N$ is a prime $\Gamma$-near-ring, $a\Gamma U = 0$ or $b = 0$.

If $b = 0$, then we are done. So if $b \neq 0$, then $a\Gamma U = 0$. Applying Lemma 2.3, $a = 0$.

(ii) Suppose $D(U)\Gamma a = \{0\}$, for $a \in N$. Then for all $u \in U$ and $b \in N$, from Lemma 2.2, we have for all $a, b \in N$ and $\alpha, \beta \in \Gamma$, $0 = D(b\alpha u)\beta a = (b\alpha D(u) + D(b)\alpha u)\beta a = b\alpha D(u)\beta a + D(b)\alpha u\beta a = D(b)\alpha u\beta a$. Hence $D(b)\Gamma U\Gamma a = \{0\}$ for all $b \in N$. Since $D$ is a nonzero derivation on $N$, we have that $a = 0$ by the statement (i).

(iii) Suppose $a\Gamma D(U) = \{0\}$ for $a \in N$. Then for all $u \in U$, $b \in N$ and $\alpha, \beta \in \Gamma$,

$0 = a\alpha D(u\beta b) = a\alpha\{u\beta D(b) + D(u)\beta b\} = a\alpha u\beta D(b) + a\alpha D(u)\beta b = a\alpha u\beta D(b)$.

Hence $a\Gamma U\Gamma D(b) = \{0\}$ for all $b \in N$. From the statement (i) and $D$ is a nonzero derivation on $N$, we have that $a = 0$.

We remark that to obtain any of the conclusions of Lemma 2.5, it is not sufficient to assume that $U$ is a right $N$-subset, even in the case that $N$ is a $\Gamma$-ring.

**Theorem 2.7**. Let $N$ be a prime $\Gamma$-near-ring and $U$ be a right $N$-subset of $N$. If $D$ is a nonzero derivation on $N$ such that $D^2(U) = 0$, then $D^2 = 0$.

**Proof**. For all $u, v \in U$ and $\alpha \in \Gamma$, we have $D^2(u\alpha v) = 0$. Then

$0 = D^2(u\alpha v) = D(D(u\alpha v)) = D\{D(u)\alpha v + u\alpha D(v)\}$

$= D^2(u)\alpha v + D(u)\alpha D(v) + D(u)\alpha D(v) + u\alpha D^2(v)$

$= D^2(u)\alpha v + 2D(u)\alpha D(v) + u\alpha D^2(v)$.

Thus $2D(u)\Gamma D(U) = \{0\}$ for all $u \in U$. From Lemma 2.5(iii), we have $2D(u) = 0$.

Now for all $b \in N$, $u \in U$ and $\alpha \in \Gamma$, $D^2(u\alpha b) = u\alpha D^2(b) + 2D(u)\alpha D(b) + D^2(u)\alpha b$. Hence $U\Gamma D^2(b) = \{0\}$ for all $b \in N$. By Lemma 2.3, we have $D^2(b) = 0$ for all $b \in N$. Consequently $D^2 = 0$.

**Lemma 2.8**. Let $D$ be a derivation of a prime $\Gamma$-near-ring $N$ and $a$ be an element of $N$. If $a\alpha D(x) = 0$ (or $D(x)\alpha a = 0$) for all $x \in N$, $\alpha \in \Gamma$, then either $a = 0$ or $D$ is zero.

**Proof**. Suppose that $aŬD(x) = 0$ for all $x \in N$, $Ŭ \in \Gamma$. Replacing $x$ by $xбy$, (for all $б \in \Gamma$) we have that $aŬD(xбy) = 0 = aŬD(x)бy + aŬxбD(y)$ by Lemma 2.2. Then $aŬxбD(y) = 0$ for all $x, y \in N$, $Ŭ, б \in \Gamma$.

If $D$ is not zero, that is, if $D(y) \neq 0$ for some $y \in N$, then, since $N$ is a prime $\Gamma$-near-ring, $a\Gamma N\Gamma D(y)$ implies that $a = 0$.

Now we prove our main result.

**Theorem 2.9**. Let $N$ be a $\Gamma$-prime near-ring with nonzero derivations $D_1$ and $D_2$ such that for all $x, y \in N$ and $Ŭ \in \Gamma$, $D_1(x)ŬD_2(y) = -D_2(x)ŬD_1(y)$ (1)

Then $N$ is an abelian $\Gamma$-near-ring.

**Proof**. Let $x, u, v \in N$, $Ŭ \in \Gamma$. From the condition (1), we obtain that

$0 = D_1(x)ŬD_2(u + v) + D_2(x)ŬD_1(u + v)$

$= D_1(x)Ǘ[D_2(u) + D_2(v)] + D_2(x)Ǘ[D_1(u) + D_1(v)]$

$= D_1(x)ŬD_2(u) + D_1(x)ŬD_2(v) + D_2(x)ŬD_1(u) + D_2(x)ŬD_1(v)$

$= D_1(x)ŬD_2(u) + D_1(x)ŬD_2(v) - D_1(x)ŬD_2(u) - D_1(x)ŬD_2(v)$

$= D_1(x)Ǘ[D_2(u) + D_2(v) - D_2(u) - D_2(v)] = D_1(x)ŬD_2(u + v - u - v)$.

Thus $D_1(N)\Gamma D_2(u + v - u - v) = \{0\}$. (2)

By Lemma 2.8, we have $D_2(u + v - u - v) = 0$. (3)

Now, we substitute $xбu$ and $xбv$ ($б \in \Gamma$) instead of $u$ and $v$ respectively in (3). Then from Lemma 2.1, we deduce that for all $x, u, v \in N$, $б \in \Gamma$,

$0 = D_2(xбu + xбv - xбu - xбv) = D_2[xб(u + v - u - v)]$

$= D_2(x)б(u + v - u - v) + xбD_2(u + v - u - v) = D_2(x)б(u + v - u - v)$.

Again, applying Lemma 2.8, we see that for all $u, v \in N$, $u + v - u - v = 0$.

Consequently, $N$ is an abelian $\Gamma$-near-ring.

**Theorem 2.10**. Let $N$ be a prime $\Gamma$-near-ring of 2-torsion free and let $D_1$ and $D_2$ be derivations with the condition $D_1(a)ŬD_2(b) = D_2(b)ŬD_1(a)$ (4)

for all $a, b \in N$ and $Ŭ \in \Gamma$ on $N$. Then $D_1D_2$ is a derivation on $N$ if and only if either $D_1 = 0$ or $D_2 = 0$.

**Proof**. Suppose that $D_1D_2$ is a derivation. Then we obtain for $Ŭ \in \Gamma$,

$D_1D_2(aŬb) = aŬD_1D_2(b) + D_1D_2(a)Ŭb$. (5)

Also, since $D_1$ and $D_2$ are derivations, we get

$D_1D_2(aŬb) = D_1(D_2(aŬb)) = D_1(aŬD_2(b) + D_2(a)Ŭb) = D_1(aŬD_2(b)) + D_1(D_2(a)Ŭb)$

$= aŬD_1D_2(b) + D_1(a)ŬD_2(b) + D_2(a)ŬD_1(b) + D_1D_2(a)Ŭb$. (6)

From (5) and (6) for $D_1D_2(aŬb)$ for all $a, b \in N$, $Ŭ \in \Gamma$, $D_1(a)ŬD_2(b) + D_2(a)ŬD_1(b) = 0$.

(7)

Hence from Theorem 2.9, we know that $N$ is an abelian $\Gamma$-near-ring.

Replacing $a$ by $a\gamma D_2(c)$ in (7), and using Lemma 2.1 and Lemma 2.2, we obtain that

$0 = D_1(a\gamma D_2(\mathrm{c}))\gamma D_2(b) + D_2(a\gamma D_2(c))\gamma D_1(b)$

$= \{D_1(a)\gamma D_2(c) + a\gamma D_1D_2(c)\}\gamma D_2(b) + \{a\gamma D_2^2(c) + D_2(a)\gamma D_2(c)\}\gamma D_1(b)$

$= D_1(a)\gamma D_2(c)\gamma D_2(b) + a\gamma D_1D_2(c)\gamma D_2(b) + a\gamma D_2^2(c)\gamma D_1(b) + D_2(a)\gamma D_2(c)\gamma D_1(b)$

$= D_1(a)\gamma D_2(c)\gamma D_2(b) + a\gamma\{D_1D_2(c)\gamma D_2(b) + D_2^2(c)\gamma D_1(b)\} + D_2(a)\gamma D_2(c)\gamma D_1(b)$:

On the other hand, replacing $a$ by $D_2(c)$ in (7), we see that

$D_1(D_2(c))\gamma D_2(b) + D_2(D_2(c))\gamma D_1(b) = 0$.

This equation implies that $a\gamma\{D_1D_2(c)\gamma D_2(b) + D_2^2(c)\gamma D_1(b)\} = 0$.

Hence, from the above last long equality, we have the following equality

$D_1(a)\gamma D_2(c)\gamma D_2(b) + D_2(a)\gamma D_2(c)\gamma D_1(b) = 0$, for all $a, b, c\in N$, $\gamma\in\Gamma$. (8)

Replacing $a$ and $b$ by $c$ in (7) respectively, we see that

$D_2(c)\gamma D_1(b) = -D_1(c)\gamma D_2(b)$, $D_1(a)\gamma D_2(c) = -D_2(a)\gamma D_1(c)$.

So that (8) becomes

$0 = \{-D_2(a)\gamma D_1(c)\}\gamma D_2(b) + D_2(a)\gamma\{-D_1(c)\gamma D_2(b)\}$

$= D_2(a)\gamma(-D_1(c))\gamma D_2(b) + D_2(a)\gamma(-D_1(c))\gamma D_2(b)$

$= D_2(a)\gamma\{(-D_1(c))\gamma D_2(b) - D_1(c)\gamma D_2(b)\}$ for all $a, b, c\in N$, $\gamma\in\Gamma$. If $D_2 \neq 0$, then by Lemma 2.8, we have the equality: $(-D_1(c))\gamma D_2(b) - D_1(c)\gamma D_2(b) = 0$,

that is, $D_1(c)\gamma D_2(b) = (-D_1(c))\gamma D_2(b)$, for all $b, c\in N$, $\gamma\in\Gamma$. (9)

Thus, using the given condition of our theorem, we get

$(-D_1(c))\gamma D_2(b) = D_1(-c)\gamma D_2(b) = D_2(b)\gamma D_1(-c) = D_2(b)\gamma(-D_1(c))$

$= -D_2(b)\gamma D_1(c) = -D_1(c)\gamma D_2(b)$. (10)

From (9) and (10) we have that, for all $b, c\in N$, $\gamma\in\Gamma$, $2D_1(c)\gamma D_2(b) = 0$.

Since $N$ is of 2-torsion free, $D_1(c)\gamma D_2(b) = 0$. Also, since $D_2$ is not zero, by Lemma 2.8, we see that $D_1(c) = 0$ for all $c\in N$. Therefore $D_1 = 0$. Consequently, either $D_1 = 0$ or $D_2 = 0$.

The converse verification is obvious. Thus our proof is complete.

As a consequence of Theorem 2.10, we get the following important statement.

**Corollary 2.11**. Let $N$ be a prime $\Gamma$-near-ring of 2-torsion free, and let $D$ be a derivation on $N$ such that $D^2 = 0$. Then $D = 0$.

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