SEMI DERIVATIONS OF PRIME GAMMA RINGS

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ABSTRACT

Let M be a prime \( \Gamma \)-ring satisfying a certain assumption (*). An additive mapping \( f : M \rightarrow M \) is a semi-derivation if \( f(x\alpha y) = f(x)\alpha g(y) + x\alpha f(y) = f(x)\alpha y + g(x)\alpha f(y) \) and \( f(g(x)) = g(f(x)) \) for all \( x, y \in M \) and \( \alpha \in \Gamma \), where \( g : M \rightarrow M \) is an associated function. In this paper, we generalize some properties of prime rings with semi-derivations to the prime \( \Gamma \)-rings with semi-derivations.

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1. Introduction


In this paper, we generalize some results of prime rings with semi-derivations to the prime \( \Gamma \)-rings with semi-derivations.

2. Preliminaries

Let \( M \) and \( \Gamma \) be additive abelian groups. \( M \) is called a \( \Gamma \)-ring if for all \( x, y, z \in M \) and \( \alpha, \beta \in \Gamma \) the following conditions are satisfied:

\[
\begin{align*}
\text{(i)} & \quad x\beta y \in M, \\
\text{(ii)} & \quad (x + y)\alpha z = x\alpha z + y\alpha z, \quad x(\alpha + \beta)y = x\alpha y + x\beta y, \quad x\alpha(y + z) = x\alpha y + x\alpha z, \\
\text{(iii)} & \quad (x\alpha y)\beta z = x\alpha(y\beta z).
\end{align*}
\]

Let \( M \) be a \( \Gamma \)-ring with center \( C(M) \). For any \( x, y \in M \), the notation \([x, y]_\alpha\) and \((x, y)_\alpha\) will denote \( x\alpha y - y\alpha x \) and \( x\alpha y + y\alpha x \) respectively. We know that \([x\beta y, z]_\alpha = x\beta[y, z]_\alpha + [x, z]_\alpha \beta y + x[\beta, \alpha]_\gamma y \) and \([x, y]_{\alpha\beta}z = y\beta[x, z]_\alpha + [x, y]_\alpha \beta z + y[\beta, \alpha]_\beta z\), for all \( x, y, z \in M \) and for
all $\alpha, \beta \in \Gamma$. We shall take an assumption (*) $x\alpha y\beta z = x\beta y\alpha z$ for all $x, y, z \in M$, $\alpha, \beta \in \Gamma$. Using the assumption (*) the identities $[x\beta y, z]_\alpha = x\beta [y, z]_\alpha + [x, z]_\alpha \beta y$ and $[x, y\beta z]_\alpha = y\beta [x, z]_\alpha + [x, y]_\alpha \beta z$, for all $x, y, z \in M$ and for all $\alpha, \beta \in \Gamma$ are used extensively in our results. So we make extensive use of the basic commutator identities: $(x\beta y, z)_\alpha = (x, z)_\alpha \beta y + x\beta [y, z]_\alpha = [x, z]_\alpha \beta y + x\beta (y, z)_\alpha$. A $\Gamma$-ring $M$ is to be n-torsion free if $nx = 0$, $x \in M$ implies $x = 0$. Recall that a $\Gamma$-ring $M$ is prime if $x\Gamma y = 0$ implies that $x = 0$ or $y = 0$.

A mapping $D$ from $M$ to $M$ is said to be commuting on $M$ if $[D(x), x]_\alpha = 0$ holds for all $x \in M$, $\alpha \in \Gamma$, and is said to be centralizing on $M$ if $[D(x), x]_\alpha \in C(M)$ holds for all $x \in M$, $\alpha \in \Gamma$. An additive mapping $D$ from $M$ to $M$ is called a derivation if $D(x\alpha y) = D(x)\alpha y + x\alpha D(y)$ holds for all $x, y \in M$, $\alpha \in \Gamma$.

Let $M$ be a $\Gamma$-ring. An additive mapping $d: M \to M$, is called a semi-derivation associated with a function $g: M \to M$, if, for all $x, y \in M$, $\alpha \in \Gamma$,

(i) $d(x\alpha y) = d(x)\alpha g(y) + x\alpha d(y) = d(x)\alpha y + g(x)\alpha d(y)$,

(ii) $d(g(x)) = g(d(x))$.

If $g = I$, i.e., an identity mapping of $M$, then all semi-derivations associated with $g$ are merely ordinary derivations. If $g$ is any endomorphism of $M$, then other examples of semi-derivations are of the form $d(x) = x - g(x)$.

**Example 2.1**

Let $M_1$ be a $\Gamma_1$-ring and $M_2$ be a $\Gamma_2$-ring. Consider $M = M_1 \times M_2$ and $\Gamma = \Gamma_1 \times \Gamma_2$.

Define addition and multiplication on $M$ and $\Gamma$ by

$(m_1, m_2) + (m_3, m_4) = (m_1 + m_3, m_2 + m_4),$

$(\alpha_1, \alpha_2) + (\alpha_3, \alpha_4) = (\alpha_1 + \alpha_3, \alpha_2 + \alpha_4),$

$(m_1, m_2)(\alpha_1, \alpha_2)(m_3, m_4) = (m_1\alpha_1 m_3, m_2\alpha_2 m_4),$

for every $(m_1, m_2), (m_3, m_4) \in M$ and $(\alpha_1, \alpha_2), (\alpha_3, \alpha_4) \in \Gamma$.

Under these addition and multiplication $M$ is a $\Gamma$-ring. Let $\delta: M_1 \to M_1$ be an additive map and $\tau: M_2 \to M_2$ be a left and right $M_1^\Gamma$-module which is not a derivation. Define $d: M \to M$ such that $d((m_1, m_2)) = (0, \tau(m_2))$ and $g: M \to M$ such that $g((m_1, m_2)) = (\delta(m_1), 0), m_1 \in M_1, m_2 \in M_2$. Then it is clear that $d$ is a semi-derivation of $M$ (with associated map $g$) which is not a derivation.

**3. Semi Derivations of Prime $\Gamma$-rings**

We obtain our results.

**Lemma 3.1**

Let $M$ be a prime $\Gamma$-ring satisfying the assumption (*) and let $m \in M$. If

$$[[m, x]_\alpha, x]_\alpha = 0$$

for all $x \in M$, $\alpha \in \Gamma$, then $x \in C(M)$. 


Proof

A linearization of 

\[ [m, x]_\alpha, x \beta = 0 \text{ for all } x \in M, \alpha \in \Gamma, \]

gives

\[ [m, x]_\alpha, x \beta + [m, y]_\alpha, x \beta = 0 \text{ for all } x, y \in M, \alpha \in \Gamma. \quad (1) \]

Replacing \( y \) by \( y \beta x \) in (1) and using \( [m, x]_\alpha, x \beta = 0 \text{ for all } x \in M, \alpha \in \Gamma, \)

we obtain

\[ 0 = [m, x]_\alpha, y \beta x + [y, x]_\alpha, \beta x + [y, x]_\alpha, [m, x]_\alpha, \text{ for all } x \in M, \alpha \in \Gamma, \]

Applying (1), we then get \( [y, x]_\alpha, \beta [m, x]_\alpha = 0, \) for all \( x, y \in M, \alpha \in \Gamma. \) Taking \( y \beta z \) for \( y \) in this relation and using \( [y, x]_\alpha, \beta [m, x]_\alpha = 0, \) for all \( x \in M, \alpha \in \Gamma. \) Since \( M \) is prime, \( [m, x]_\alpha = 0. \) This implies \( x \in C(M). \)

Theorem 3.2

Let \( M \) be a non-commutative 2-torsion free prime \( \Gamma \)-ring satisfying the condition (*) and \( d \) is a semi-derivation of \( M \) with \( g: M \rightarrow M \) is an onto endomorphism. If the mapping \( x \rightarrow [a \beta d(x), x]_\alpha \) for all \( \alpha, \beta \in \Gamma, \) is commuting on \( M, \) then \( a = 0 \) or \( d = 0. \)

Proof

Firstly, we assume that \( a \) be a nonzero element of \( M. \) Then we know that the mapping \( x \rightarrow [a \beta d(x), x]_\alpha \) is commuting on \( M. \) Thus we have \( [[a \beta d(x), x]_\alpha, x]_\alpha = 0. \) By lemma 3.1, we have

\[ [a \beta d(x), x]_\alpha = 0, \text{ for all } x \in M, \alpha, \beta \in \Gamma. \quad (2) \]

By linearizing (2), we have \( [a \beta d(x), y]_\alpha + [a \beta d(y), x]_\alpha = 0, \text{ for all } x, y \in M, \alpha, \beta \in \Gamma. \) (3)

From this relation it follows that

\[ a\beta [d(x), y]_\alpha + [a, y]_\alpha \beta [d(x), x]_\alpha + [a, x]_\alpha \beta [d(y), x]_\alpha + [a \beta d(d(y)), x]_\alpha = 0, \text{ for all } x, y \in M, \alpha, \beta \in \Gamma. \quad (4) \]

Replacing \( y \) by \( y \delta \) in (3) and using (2), we get \( [a \beta d(x), y \delta x]_\alpha + [a \beta d(y \delta x), x]_\alpha = 0, \) for all \( x, y \in M, \alpha, \beta \in \Gamma. \) We get

\begin{align*}
  &y\delta [a \beta d(x), x]_\alpha + [a \beta d(x), y]_\alpha \delta x + [a \beta d(y \delta x), g(y) \delta d(x)], x]_\alpha \\
  &\quad = [a \beta d(x), y]_\alpha \delta x + [a \beta d(y \delta x), x]_\alpha + [a \beta g(y) \delta d(x), x]_\alpha \\
  &\quad = a \beta [d(x), y]_\alpha \delta x + [a, y]_\alpha \beta [d(x), x]_\alpha + a \beta [d(y \delta x), x]_\alpha + [a \beta d(y), x]_\alpha \delta x + [a \beta g(y) \delta d(x)], x]_\alpha \\
  &\quad = a \beta [d(x), y]_\alpha \delta x + [a, y]_\alpha \beta [d(x), x]_\alpha + a \beta [d(y), x]_\alpha \delta x + [a, x]_\alpha \beta [d(y \delta x) + a \beta g(y) \delta d(x), x]_\alpha + a \beta [g(y), x]_\alpha \delta d(x) + [a, x]_\alpha \beta [g(y \delta d(x) = 0 \text{ for all } x, y \in M, \alpha, \beta \in \Gamma. \quad (5)\end{align*}
Right multiplication of (3) by $\delta x$ gives
\[ a\beta[d(x), y]_\alpha \delta x + [a, y]_\alpha b d(x) \delta x + a\beta[d(y), x]_\alpha \delta x + [a, x]_\alpha b d(y) \delta x = 0, \text{ for all } x, y \in M, \alpha, \beta \in \Gamma. \quad (6) \]

Subtracting (6) from (5), we obtain
\[ a\beta g(y) \delta[d(x), x]_\alpha + a\beta[g(y), x]_\alpha \delta d(x) = 0 \text{ for all } x, y \in M, \alpha, \beta \in \Gamma. \quad (7) \]

Taking $a\lambda g(y)$ instead of $g(y)$ in (7), we have
\[ a\beta a\lambda g(y) \delta[d(x), x]_\alpha + a\beta[a\lambda g(y), x]_\alpha \delta d(x) + [a, x]_\alpha \lambda g(y) \delta d(x) = 0 \text{ for all } x, y \in M, \alpha, \beta, \lambda, \delta \in \Gamma. \quad (8) \]

Left multiplication of (6) by $a\lambda$ leads to
\[ a\lambda a\beta g(y) \delta[d(x), x]_\alpha + a\lambda a\beta g(y) \delta d(x) = a\beta a\lambda g(y) \delta[d(x), x]_\alpha + a\beta[a\lambda g(y), x]_\alpha \delta d(x) = 0 \text{ for all } x, y \in M, \alpha, \beta, \lambda, \delta \in \Gamma. \quad (9) \]

Subtracting (9) from (8), we get
\[ [a, x]_\alpha \beta a\lambda g(y) \delta d(x) = 0 \text{ for all } x, y \in M, \alpha, \beta, \lambda, \delta \in \Gamma. \]

Since $M$ is prime, we obtain that for any $x \in M$ either $d(x) = 0$ or $[a, x]_\alpha = 0$.

It means that $M$ is the union of its additive subgroups $P = \{x \in M: d(x) = 0\}$ and $Q = \{x \in M: [a, x]_\alpha = 0\}$. Since a group cannot be the union of two proper subgroups, we find that either $P = M$ or $Q = M$.

If $P = M$, then $d = 0$. If $Q = M$, then this implies that $[a, x]_\alpha = 0$, for all $x \in M, \alpha, \beta \in \Gamma$.

Let us take $x \delta y$ instead of $x$ in this relation. Then we get
\[ [a, x \delta y]_\alpha \beta a = 0, \text{ for all } x, y \in M, \alpha, \beta \in \Gamma. \]

We get
\[ [a, x \delta y]_\alpha \beta a = x \delta[a, y]_\alpha \beta a + [a, x]_\alpha \delta y \beta a = [a, x]_\alpha \delta y \beta a = 0, \text{ for all } x, y \in M, \alpha, \beta, \delta \in \Gamma. \]

Since $a \in M$ is nonzero and $M$ is prime, we obtain $a \in C(M)$. Thus by this and (2), the relation (7) reduces to
\[ a\beta[g(y), x]_\alpha \delta d(x) = 0, \text{ for all } x, y \in M, \alpha, \beta, \delta \in \Gamma. \]

Since $g$ is onto, we see that $a\beta z y u, x]_\alpha \delta d(x) = z \beta u y [u, x]_\alpha \delta d(x) = 0$, for all $x, u, z \in M, \alpha, \beta, \delta, \gamma \in \Gamma$. Now by primeness of $M$, we obtain that $[u, x]_\alpha \delta d(x) = 0$, for all $x, u \in M, \alpha, \beta, \delta \in \Gamma$. 


Replacing \( u \) by \( u\lambda w \), we get \([u, x]_\alpha \lambda w \delta d(x) = 0\), for all \( x, u, w \in M, \alpha, \beta, \delta, \lambda \in \Gamma \). By the primeness of \( M \), \([u, x]_\alpha = 0 \) or \( d(x) = 0 \). Again using the fact that a group cannot be the union of two proper subgroups, it follows that \( d = 0 \), since \( M \) is non-commutative, i.e., \([u, x]_\alpha \). Hence we see that, in any case, \( d = 0 \). This completes the proof.

**Theorem 3.3**

Let \( M \) be a prime 2-torsion free \( \Gamma \)-ring satisfying the condition (*), \( d \) is a nonzero semi-derivation of \( M \), with associated endomorphism \( g \) and \( a \in M \). If \( g \neq \pm I \) (I is an identity map of \( M \)), then \((d(M), a)_\alpha = 0\) if and only if \( d((M, a)_\alpha) = 0\).

**Proof**

Suppose \((d(M), a)_\alpha = 0\). Firstly, we will prove that \( d(a) = 0 \). If \( a = 0 \) then \( d(a) = 0 \). So we assume that \( a \neq 0 \). By our hypothesis, we have \((d(x), a)_\alpha = 0\), for all \( x \in M, \alpha \in \Gamma \).

From this relation, we get

\[
0 = (dx\beta a, a)_\alpha = (dx\beta g(a) + x\beta d(a), a)_\alpha
\]

\[
= d(x)\beta [g(a), a]_\alpha + (d(x), a)\alpha \beta a + x\beta (d(a), a)_\alpha + [x, a]_\alpha \beta d(a),
\]

and so, \([x, a]_\alpha \beta d(a) = 0\), for all \( x \in M, \alpha, \beta \in \Gamma \). (10)

Now, replacing \( x \) by \( x\delta y \) in (10), we get

\[
[x\delta y, a]_\alpha \beta d(a) = 0, \text{ for all } x \in M, \alpha, \beta \in \Gamma.
\]

By calculation we get,

\[
x\delta y[a]_\alpha \beta d(a) + [x, a]_\alpha \delta y \beta d(a) = [x, a]_\alpha \delta y \beta d(a) = 0, \text{ for all } x \in M, \alpha, \beta, \delta \in \Gamma
\]

(11)

The primeness of \( M \) implies that \([x, a]_\alpha = 0 \) or \( d(a) = 0 \) that is, \( a \in C(M) \) or \( d(a) = 0 \).

Now suppose that \( a \in C(M) \). Since \((d(a), a)_\alpha = 0 \), we have \( d(a)\alpha a + a\alpha d(a) = 2a\alpha d(a) = 0 \). Since \( M \) is 2-torsion free, \( a\alpha d(a) = 0 \). Since we assumed that \( 0 \neq a \) and \( M \) is a prime \( \Gamma \)-ring, we get \( d(a) = 0 \). Hence we have \((d(x, a)_\alpha)

\[
= d(x\alpha a + a\alpha x) = d(x\alpha a) + d(a\alpha x) = d(x)\alpha a + g(x\alpha) d(a) + d(a)\alpha g(x) + a\alpha d(x)
\]

\[
= (g(x), d(a))_\alpha + (d(x), a)_\alpha = (d(x), a)_\alpha = 0, \text{ for all } x \in M, \alpha \in \Gamma.
\]

(12)

Hence \((d(x), a)_\alpha = 0 \).

Conversely, for all \( x \in M, \)

\[
0 = d((a\beta x, a)_\alpha) = d(a\beta(x, a)_\alpha + [a, a]_\alpha \beta x) = d(a\beta(x, a)_\alpha) = d(a)\beta(x, a)_\alpha + g(a)\beta d((x, a)_\alpha).
\]

We have

\[
d(a)\beta(x, a)_\alpha = 0, \text{ for all } x \in M, \alpha, \beta \in \Gamma.
\]

Replacing \( x \) by \( x\delta y \) in (12), we get

\[
0 = d(a)\beta(x\delta y, a)_\alpha = d(a)\beta x\delta[y, a]_\alpha + d(a)\beta(x, a)_\alpha \delta y = d(a)\beta x\delta[y, a]_\alpha.
\]

This implies that \( d(a)\beta x\delta[x, a]_\alpha = 0, \text{ for all } x \in M, \alpha, \beta, \delta \in \Gamma \).
For the primeness of $M$, we have either $d(a) = 0$ or $a \in C(M)$. If $d(a) = 0$, then we have $0 = d((x), a)_{\alpha} = (d(x), a)_{\alpha} + (d(a), g(x))_{\alpha} = (d(x), a)_{\alpha}$, for all $x \in M, \alpha \in \Gamma$.

This yields that $(d(x), a)_{\alpha} = 0$. If $a \in C(M)$, then we have $0 = d((a, a)_{\alpha}) = 2d(a)_{\alpha}(a + g(a))$. Since $M$ is 2-torsion free, we obtain $d(a)\alpha(a + g(a)) = 0$. Since $M$ is prime we have $d(a) = 0$ or $a + g(a) = 0$. But since $g$ is different from $\neq \pm 1$, we find that $d(a) = 0$. Finally, $(d(x), a)_{\alpha} = 0$ implies the required result.

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