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Dynamical Properties of a Stage Structured Model of a Species with Michaelis-Menten Type Harvesting and Intra-species Competition

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Abstract

The study of stage-structured models in population dynamics is an effective way to understand future demographics better. This article investigates a single-species stage-structured model for its dynamical properties with an intra-group competition in the juvenile species and with Michaelis-Menten type harvesting on the adult group. The study focuses on both qualitative and numerical investigations. The dynamics of the model remain complex and rich. We tried to understand the biological, ecological, and economic implications of the dynamic data found from the investigation. Suitable Lyapunov and Dulac functions have been used for the global asymptotic stability of equilibrium points. The global asymptotic stability of internal equilibrium points reveals the possibility of bistability under reasonable parameter values. Despite intraspecies competition in the juvenile group, the adult population may be harvested significantly, ensuring the sustainable existence of the population.

Keywords: Stage Structured Model, Harvesting, Michaelis-Menten form of harvesting, Local and Global Stabilities.

I. Introduction

A harvesting model is a population growth model of a species from which a portion of the population is harvested at regular or irregular intervals. In harvesting models, a negative term called a harvesting function is added in the expression of the growth rate that represents the portion of the population that is harvested. The two most common harvesting functions are the constant function and the linear function[1]. While these two choices are obviously simple, they do not give us results that are very practical, as it is the extent of balance between long term progression of the ecosystem, the economic benefits, and the sustainability of the ecosystem that is the primary indicator of a practical model, and those two choices fail to ensure such a balance[2,3]. These shortcomings can be rectified significantly by considering non-linear functions[4,5,6]. The Michaelis-Menten type harvesting model turned out to be the most practical from both a biological and an overall economic standpoint when compared to other non-linear harvesting strategies[7]. Non-linear functions of the form $h(x) = \frac{ax}{b+cx}$ are used to realize harvesting methods of the Michaelis-Menten type, given certain appropriate parameters $a, b, and\ c$. These non-linear harvesting models exhibit more complex and practically significant dynamical features when compared to the model with constant and linear harvesting. Hu et al. investigated a prey-predator model featuring rich bifurcation-exhibiting Michaelis-Menten-type harvesting techniques[8].

Many authors, while studying harvesting models, have considered age-structured models. The study of stage-structured models is motivated by the fact that a species generally shows different characteristics at different stages (juvenile, adult, or old adult) of its life cycle, producing differences in the behavior of their dynamics. There have been many studies that involve age-structured population growth models, with important findings regarding sustainability, coexistence, and danger of extinction. One such study was done by Lei, whose work was based on the following commensalism model for two species[9].

$$\begin{cases} \frac{du_1}{dt} = \alpha u_2 - \beta u_1 - \delta_1 u_1 \\ \frac{du_2}{dt} = \beta u_1 - \delta_2 u_2 - \gamma u_2^2 + d u_2 v \\ \frac{dv}{dt} = v(b_2 - a_2 v) \end{cases} \tag{1.1}$$

where u_1, u_2 denote densities of the juvenile and adult populations at any time t of the first species, v denotes the density of the second species, and the parameters $\alpha, \beta, \delta_1, \delta_2, d, a_2, and\ b_2$ are all positive. While studying the stability of the equilibrium points, it was found that strong inter-species cooperation is one of the best ways to prevent the extinction of endangered species. Ma, Li, et al. then proposed the stage-structure-prey-predator model in the delay differential equation for two species, considering that the juvenile species requires time to mature[10,11,12,13].

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$$\begin{cases} \frac{dx_1}{dt} = r_1(t)x_1(t) - d_{11}x_1(t) - r_1(t - \tau_1)e^{-d_{11}\tau_1 x_2(t-\tau_1)} \\ \frac{dx_2}{dt} = r_1(t - \tau_1)e^{-d_{11}\tau_1 x_2(t-\tau_1)} - d_{12}x_2(t) - b_1(t)x_2^2(t) - c_1(t)x_2(t)y_2(t) \\ \frac{dy_1}{dt} = r_2(t)y_2(t) - \mathrm{d}_{22}\mathrm{y}_1(t) - \mathrm{r}_2(t - \tau_2)e^{-d_{22}\tau_2 y_2(t-\tau_2)} \\ \frac{dy_2}{dt} = r_2(t - \tau_2)e^{-d_{22}\tau_2 y_2(t-\tau_2)} - d_{21}y_2(t) - b_2(t)y_2^2(t) - c_2(t)x_2(t)y_2(t) \end{cases} \tag{1.2}$$

In their study of the nature of stabilities of the stationary solutions, conditions of extinction, and persistence of the species, it was found that the predator population may survive even in the absence of the prey if the parameter values satisfy some given conditions, which just gave a pathway for the system to be consistent under some right conditions.

Harvesting strategies have been integrated with age-structured models by many authors. As an example, the following single-species age-structured model with a linear harvesting technique for both adult and juvenile populations was explored by Lei and Xiao[2]. The model's dynamics, as it turns out, are relatively straightforward, consisting of only two cases: a boundary equilibrium $(0,0)$ and an internal equilibrium (x^*, y^*), both globally asymptotically stable under two different sets of parameter values.

$$\begin{cases} \frac{dx}{dt} = \alpha y - \beta x - \delta_1 x - q_1 Emx \\ \frac{dy}{dt} = \beta x - \delta_2 y - \Upsilon y^2 - q_1 Emy \end{cases} \tag{1.3}$$

Numerous academics investigated age-structured models that included Michaelis-Menten-type harvesting procedures. Liu et al.[14], for example, examined the following amensalism model using cover harvesting only from the juvenile population.

$$\begin{cases} \frac{dx}{dt} = a_1 x - b_1 x^2 - c_1(1-k)xy - \frac{qE(1-k)x}{m_1 E + m_2(1-k)} \\ \frac{dy}{dt} = a_2 y - b_2 y^2 \end{cases} \tag{1.4}$$

According to the study, rich bifurcation phenomena, such as saddle and transcritical bifurcations, are displayed by this model. Finding the threshold value for the best harvesting under specific parameter values is obtainable.

Yu, Zhu, and Chen considered harvesting the juvenile population in Michaelis-Menten form in their work on the stage-structured model below[15]. They observed practically useful dynamics.

$$\begin{cases} \frac{dx}{dt} = \alpha y - \beta x - \delta_1 x - \frac{hEx}{mE + nx} \\ \frac{dy}{dt} = \beta x - \delta_2 y - \gamma y^2 \end{cases} \tag{1.5}$$

Since only adult populations are often harvested, Zhu Lia et al.[5] suggested and investigated the following model using a Michaelis-Menten kind of harvesting function for the adult population from the perspective of fisheries production. Under certain constraints on the model parameters, an internally stable boundary and a globally stable boundary were found to exist. They also exhibited exciting bifurcation phenomena.

$$\begin{cases} \frac{dx}{dt} = \alpha y - \beta x - \delta_1 x \\ \frac{dy}{dt} = \beta x - \delta_2 y - \gamma y^2 - \frac{hEy}{mE + ny} \end{cases} \tag{1.6}$$

Saima Akter et al. studied the dynamical properties of a single-species stage-structured model with Michaelis-Menten type harvesting on the adult population and linear harvesting on the juvenile population[16]. As far as we know, a single-species stage-structured model with intra-species competition in the juvenile group and Michaelis-Menten-type harvesting tactics in the adult group has not been examined by any author. The stage-structured model of a single species that we suggest and examine is shown below. This is a symmetric variant of the model (1.5). Consider the following proposed model

$$\begin{cases} \frac{dx}{dt} = \alpha y - \beta x - \delta_1 x - \delta_3 x^2 \\ \frac{dy}{dt} = \beta x - \delta_2 y - \frac{hEy}{mE + ny} \end{cases} \tag{1.7}$$

where $x, and\ y$ denotes the densities of the juvenile and adult groups of population respectively, α denotes the rate of birth in the whole population, β is the survival rate of juvenile population to adulthood, δ_1, and δ_2 denotes their death rates, δ_3 is the intra-specific competition coefficients in the Juvenile population, h is the coefficient of catchability, E stands for the external effort required to gather the adult population. All parameters including m, n are assumed to be non-negative. To make the system (1.7) dimensionless and simplified, we consider the following transformations

$$\bar{x} = \frac{\delta_3}{\beta}x, \bar{y} = \frac{\delta_3}{\beta}y, \bar{t} = \beta t$$

which transforms the system into (ignoring the bars on x, and y)

$$\begin{cases} \frac{dx}{dt} = ey - x(x + \left(1 + \frac{\delta_1}{\beta}\right)) \\ \frac{dy}{dt} = x - ay - \frac{by}{c+y} \end{cases} \tag{1.8}$$

where, $a = \frac{\delta_2}{\beta}$, $b = \frac{hE\delta_3}{n\beta^2}$, $c = \frac{mE\delta_3}{n\beta}$, $e = \alpha\beta$. If we consider the death rate of the juvenile population is negligible

($\delta_1 \ll \beta$) compare to the survival rate, the system reduces to

$$\begin{cases} \frac{dx}{dt} = ey - x(x+1) \\ \frac{dy}{dt} = x - ay - \frac{by}{c+y} \end{cases}$$

Since a number of population can only be non-negative, we consider $x \geq 0$, and $y \geq 0$, with any arbitrary initial conditions $x(0) = x_0 > 0$, x(0), and $y(0) = y_0 >$.

II. Stability Analysis of Equilibria

Boundary Equilibria

Notice that $E0(0,0)$ is the unique boundary equilibrium of the system (1.8).

The Jacobian matrix for the system (1.8) is

$$J = \begin{pmatrix} -2x-1 & e \\ 1 & -a - \frac{bc}{(c+y)^2} \end{pmatrix}$$

The determinant and the trace of Jacobian matrix are

$$Det\,J = (1+2x)\left(a + \frac{bc}{(c+y)^2}\right) - e \tag{2.1}$$

And

$$tr\,J = -\left(a + 1 + 2x + \frac{bc}{(c+y)^2}\right) < 0 \tag{2.2}$$

When $Det\,J \neq 0$, then the equilibrium point is of an elementary type[17]. When $Det\,J < 0$, then the equilibrium point is a saddle point, and the equilibrium point is degenerate when $Det\,J = 0$[18]. Notice that $Det\,J$ vanishes at $E0(0,0)$ when

$$a + \frac{b}{c} = e \tag{2.3}$$

Lemma 1. *Suppose $O(0,0)$ is an isolated stationary point of the system*[19]

$$\begin{cases} \frac{dx}{dt} = P_2(x,y) \\ \frac{dy}{dt} = y + Q_2(x,y) \end{cases} \tag{2.4}$$

where P_2 and Q_2 are analytic functions of degree at least two in a neighborhood $S_\delta(0)$. In a sufficiently small δ −neighborhood, there exists an analytic function $\phi(x)$ such that

$$\phi(x) + P_2(x, \phi(x)) = 0, |x| < \delta$$

Let $\psi(x) = (P_2(x), \phi(x)) = a_m x^m + [x]_{m+1}$, *where $a_m \neq 0, m \geq 2$, then*

i. If m is odd and $a_m > 0$, then $O(0,0)$ is an unstable node.

ii. If m is odd and $a_m < 0$, then $O(0,0)$ is a saddle.

iii. If m is even, $O(0,0)$ is a saddle node.

The stability of the boundary equilibrium $E0(0,0)$ can be determined from the trace and determinant of the Jacobian matrix $J(0,0)$.

Theorem 1. *For positive values of the parameters, $E0 = (0,0)$ is the unique boundary equilibria point of the system (1.8), for which*

i. If $b < c(e-a)$, then $E0$ is a saddle point.

ii. If $b > c(e-a)$, then $E0$ is a stable node.

iii. If $b = c(e-a)$, then $E0$ is a saddle node.

Proof. At the boundary equilibrium point (0,0), we get from (1.8),

$$Det\,J(E0) = a + \frac{b}{c} - e$$

Notice that $trJ(E0) < 0$. Therefore, $E0$ is a stable node whenever $Det\,J(E0) > 0$, i.e, $a + \frac{b}{c} > e$.

$E0$ is a saddle whenever $Det\,J(E0) < 0$, i.e., $a + \frac{b}{c} < e$. Moreover, $E0$ is becomes degenerate whenever $Det\,J(E0) = 0$, i.e, $a + \frac{b}{c} = e$. We introduce the transformation $d\tau = \frac{dt}{f+y}$ for further simplification and to capture the nature of the boundary equilibrium $E0$. Then the system (1.8) reduces to

$$\frac{dx}{d\tau} = -cx - cx^2 + cey - xy - x^2y + ey^2$$

$$\frac{dy}{d\tau} = cx - by - acy + xy - ay^2$$

We transform as follows

$$\begin{bmatrix} u \\ v \end{bmatrix} = \frac{1}{\alpha k - \beta y} \begin{bmatrix} \kappa x - \beta y \\ -\gamma x + \alpha y \end{bmatrix}.$$

Then we obtain $u = \frac{1}{p}(\kappa x - \beta y)$ and $v = \frac{1}{p}(-\gamma x + \alpha y)$, where $p = \alpha\kappa - \beta\gamma$ and the values of α, β, γ, and κ are

$\alpha = constant,$

$$\beta = \frac{a + \frac{b}{c}}{\left(1 + a + \frac{b}{c}\right)^2 c\alpha}$$

$\kappa = -\frac{b+ac}{(b+c+ac)^2\alpha}$, and

$$\gamma = \frac{\alpha}{a + \frac{b}{c}}$$

Differentiating u and v, we get

$$u' = \frac{1}{p}(\kappa x' - \beta y'), v' = \frac{1}{p}(-\gamma x' + \alpha y')$$

We compute $\frac{du}{d\tau}$ and $\frac{dv}{d\tau}$ in terms of u and v by substituting the values of $e = a + \frac{ad}{f}$, κ, β, and γ, and make the transformation $\frac{ds}{d\tau} = \alpha\kappa - \gamma\beta$. We get,

$$\frac{du}{ds} = g_{01}u^3 + g_{02}u^2 + g_{03}uv + g_{04}v^2 + r_1(u,v) \quad (2.5)$$

where, $g_{01} = \frac{b\alpha^2 + ac\alpha^2}{\left(a+\frac{b}{c}\right)(b+c+ac)^2}$.

We omit g_{02}, g_{03}, g_{04}, and $r_1(u,v)$ as they are not used further. Next, the value of $\frac{dv}{ds}$ is

$$\frac{dv}{ds} = h_{01}v + h_{02}u^2 + h_{03}v^2 + h_{04}uv + h_{05}u^3 + r_2(u,v) \quad (2.6)$$

where

$$h_{01} = \frac{a}{\left(1+a+\frac{b}{c}\right)^2} + \frac{a}{\left(a+\frac{b}{c}\right)\left(1+a+\frac{b}{c}\right)^2} + \frac{b}{\left(1+a+\frac{b}{c}\right)^2 c} + \cdots,$$

and

$$h_{02} = \frac{a\alpha^3}{\left(a+\frac{b}{c}\right)^3} + \frac{\alpha^3 - a\alpha^3 + \alpha^4}{\left(a+\frac{b}{c}\right)^2} + \frac{\alpha^3 + c\alpha^3}{a+\frac{b}{c}} - \frac{b\alpha^3}{\left(a+\frac{b}{c}\right)^3 c}. \quad (2.7)$$

We omit h_{03}, h_{04}, h_{05}, and $r_2(u,v)$ as they are not used further.

Define $H_1(u,v) = h_{01}v + h_{02}u^2 + h_{03}v^2 + h_{04}uv + h_{05}u^3 + r_4(u,v)$. Since $\frac{\partial H_1}{\partial v} \neq 0$, by implicit function theorem, there exists a function $v = w_1(u)$, such that $w(0) = 0$ and $q\big(u, w(u)\big) = 0$. The function $v = w(u)$ may be approximated iterative as follows

$$v_1 = w_1(u) = 0$$

$$v_2 = -t_{02}u^2 - t_{05}u^3,$$

$$v_3 = v_2 - H_1(u, v_2) = -h_{02}u^2 - h_{05}u^3 - (-h_{01}h_{02}u^2 - h_{01}h_{05}u^3 + h_{02}u^2) + h_{03}(-h_{02}u^2 - h_{05}u^3)^2$$

$$= h_{01}h_{02}u^2 + (h_{01}h_{05} - h_{05})u^3$$

Therefore, we have,

$$v = w_1(u) = h_{01}h_{02}u^2 + (h_{01}h_{05} - h_{05})u^3 + \cdots$$

We substitute this value of v into the equation (2.5) to get

$$\frac{du}{ds} = g_{01}u^3 + g_{02}u^2 + g_{03}uv + g_{04}v^2 + r_3(u,v)$$

$$= g_{01}u^3 + g_{02}u^2 + g_{03}u(h_{01}h_{02}u^2 + (h_{01}h_{05} - h_{05})u^3)$$

$$= g_{01}u^3 + g_{02}u^2 + g_{03}h_{01}h_{02}u^3$$

$$= g_{01}u^2 + (g_{03}h_{01}h_{02} + g_{01})u^3$$

From Lemma 1, if $g_{01} \neq 0$, i.e., $\frac{b\alpha^2 + ac\alpha^2}{\left(a+\frac{b}{c}\right)(b+c+ac)^2} \neq 0$, then $m = 2$, hence by Lemma 1, we conclude that $E0$ is a saddle node. However, due to the transformation,

$ds = (\alpha\kappa - \gamma\beta)d\tau$, then it turns into the opposite direction, and $E0$ becomes a stable node.

Internal Equilibria

Our next target is to analyze the internal equilibrium of the system (1.8). Stationary points are the solution of the system

$$\begin{cases} ey - x(x+1) = 0 \\ x - ay - \frac{by}{c+y} = 0 \end{cases} \quad (2.8)$$

From (2.8), we get from the first equation $y = \frac{x(x+1)}{e}$. Substituting y in the second equation of (2.8), and since $y \neq 0$, we get

$$ax^3 + (2a-e)x^2 + (e(ac+b) + a - e)x + e(ac+b) - ce^2 = 0 \quad (2.9)$$

To understand the nature of internal equilibria, let

$$f(x) = ax^3 + (2a-e)x^2 + (e(ac+b) + a - e)x + e(ac+b) - ce^2$$

Therefore, $f'(x) = 3ax^2 + 2(2a-e)x + (e(ac+b) + a - e)$

From $f(x) = 0$, we get

$$b = -\frac{ax^3 + (2a-e)x^2 + (ace+a-e)x + ce(a-e)}{e(1+x)} \quad (2.10)$$

By using (2.10) in (2.1), we get

$$DetJ(E) = \frac{x(x^2+x+ce)(2ax^3 + (5a-e)x^2 + (4a-2e)x + ce^2 + a - e)}{e(1+x)}$$

$$= (x^2 + x + ce)\big(xf'(x) - f(x)\big) = x(x^2 + x + ce)f'(x) \quad (2.11)$$

Let $\bar{A} = (2a-e)^2 - 3a(e(ac+b) + a - e)$, and

$$\Delta(b) = \frac{1}{9a^2}(-4(\bar{A})^3 + (3(a+e)\bar{A} - (a+e)^3 + 27a^2e^2c)^2).$$

The following cases need to be considered.

Case I: $ce - ca > b$

Theorem 2. *For $ce - ca > b$, the system (1.8) may have at most three internal equilibria*[20,21].

i. *If $\Delta(b) < 0$, all three internal equilibria are distinct. One of them is a saddle point, and the other two are stable nodes.*

ii. *If $\Delta(b) = 0$, we have two cases to consider:*

(a) *When $\bar{A} > 0$, there is one degenerate equilibrium point and a stable node.*

(b) *When $\bar{A} = 0$, there is only one degenerate internal equilibrium point.*

iii. *If $\Delta(b) > 0$, there is a unique internal stable node equilibrium point.*

Graphical Analysis for Case (I)

The analysis above and Figure 1 demonstrate that if $b < c(e-a)$, the boundary equilibrium $E0$ of the (1.8) is unstable. To verify the result, we consider the parameter values $a = 1, b = 1, c = 1/4$, and $e = \frac{496}{97}$, satisfying $c(e-a) = 1.02835 > b$ and $\Delta(b) < 0$. In this case, the system (1.8) has three internal equilibria in which $E1(0.070213,\ 0.014695)$ and $E3(2.0222,\ 1.1952)$ are stable nodes, and $E2(1.021,\ 0.40355)$ is a saddle point. The red line (stable manifold) in Figure 1 separates the first quadrant into the lower and upper areas. All solutions tend to the stable node $E3$ if the initial conditions are in the higher region, while all solutions tend to the stable node $E1$ when the initial conditions are in the lower region. As a result, juvenile and adult species are likely to coexist. This is known as bistability phenomenon.

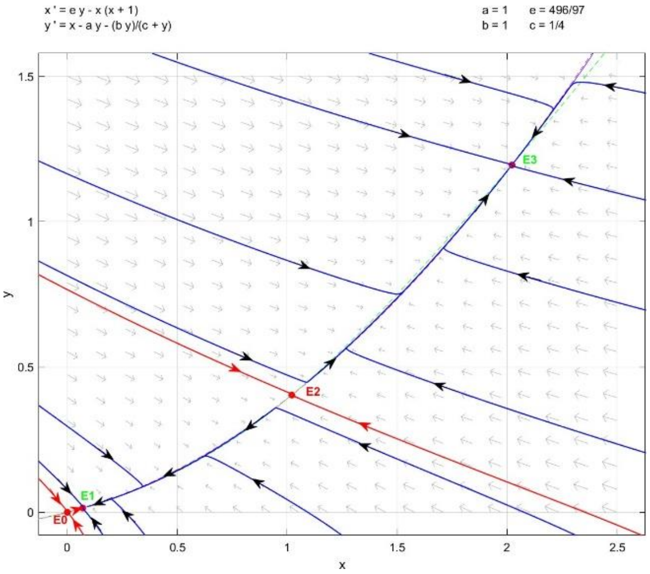


Fig. 1. The coexistence of juvenile and adult species whenever $b < c(e-a)$ and $\Delta(b) < 0$. $E1$ and $E3$ are stable nodes and $E0$ and $E2$ are saddles.

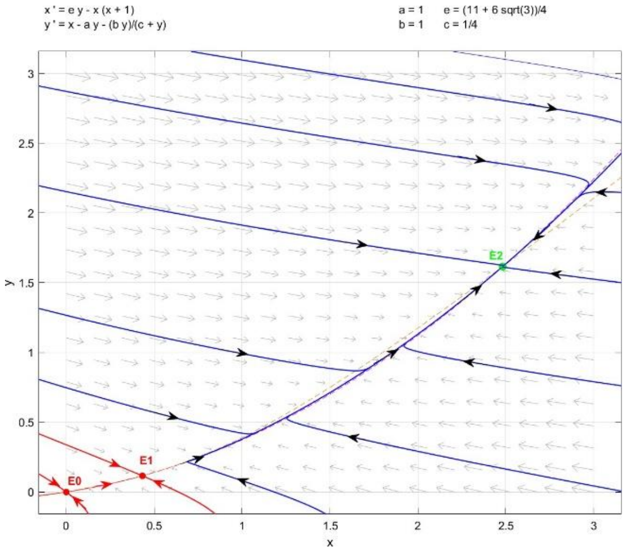


Fig. 2. The coexistence of juvenile and adult species when $b < c(e-a)$, $\Delta(b) = 0$, $\bar{A} > 0$. Two positive equilibria $E1$ (saddle) and $E2$ (stable).

If we take the parameter values $a = 1, b = 1, c = \frac{1}{4}$, and $e = \frac{1}{4}(11 + 6\sqrt{3})$, we get $c(e-a) = 1.08702 > b$, $\Delta(b) = 0$, $\bar{A} = 4.19856 > 0$. In this case, the model (1.8) has two internal equilibria, $E1$ and $E2$, which are saddle and stable nodes, respectively, and an unstable boundary equilibrium $E0$ (figure 2).

Parameter values $a = 1, b = 1, c = \frac{1}{16}$, and $e = 8$ give us $c(e-a) = 0.4375 > b$ and $\Delta(b) = 9143.51 > 0$. As a result, the system (1.8) has an unstable boundary equilibrium point $E0$ and a stable internal equilibrium point $E1$ (figure 3).

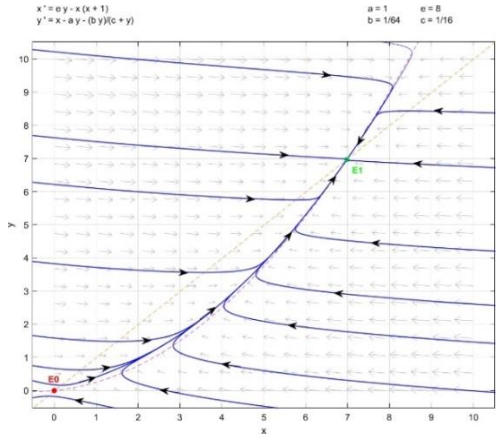


Fig. 3. The coexistence of juvenile and adult species whenever $b < c(e-a)$, and $\Delta(b) > 0$, $E1$ is an internal stable equilibrium point.

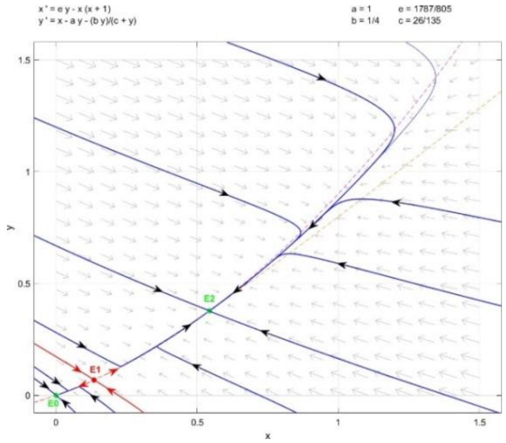


Fig. 4. $b \geq c(e-a)$, Δb <0; $E0$ and $E2$ are stable nodes and $E1$ is a saddle point.

Case II: $c(e-a) \leq b$

Theorem 3. *If* $c(e-a) \leq b$, *the system* (1.8) *may have two internal equilibria at most*[20,21]. *We have the following cases to consider,*

1. *If* $\Delta(b) < 0$, *the model* (1.8) *has two internal distinct equilibria* $E1$ *and* $E2$ *that are a saddle and a stable node, respectively. If* $\Delta(b) = 0$ *and* $\bar{A} > 0$.

i. *There is a unique degenerate internal equilibrium point* $E1$.

ii. *There is no internal equilibrium point.*

2. *When* $\Delta(b) > 0$ *or* $\Delta(b) = \bar{A} = 0$ *the system* (1.8) *has no internal equilibrium.*

2.4 Graphical Analysis for Case (II)

The analysis above and figure 4, show the existence of two internal equilibria. Depending on the initial conditions, two species will coexist permanently or will be driven into extinction. For verification of these results, we assume $a = 1, b = 1/4, c = 26/135$, and $e = 1787/805$, so that $c(e-a) = 0.234939 < b$ and $\Delta b = -0.17663$. In this case, we have a boundary equilibrium point $E0(0{,}0)$, two internal equilibria $E1(0.13434,\ 0.068649)$ and $E2(0.54346,\ 0.37786)$, which are saddle and stable, respectively.

In figure 4, the red line (stable manifold) demarcates the first quadrant into two regions, a lower region and an upper region. Any initial condition in the upper region will tend to the stable node $E2$. The two populations will coexist permanently from a biological perspective. However, the initial conditions in the lower zone will tend toward the boundary equilibrium $E0$, implying that juvenile and adult species will ultimately become extinct. Here we observe another bistability phenomenon.

To verify the case of $\Delta b = 0$ and $\bar{A} > 0$, we choose $a = \frac{2529}{4003}, b = 1, c = \frac{4003}{10116}, e = \frac{12645}{4003}$, then the model (1.8) has a distinct internal degenerate equilibrium point $E1$ (figure 5).

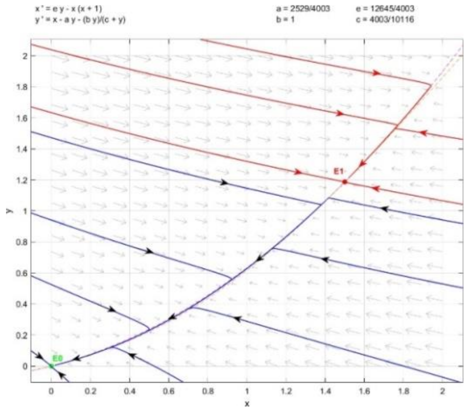


Fig. 5. The only internal equilibrium $E1$ when $b \geq c(e-a)$, $\Delta b = 0, \bar{A} > 0$.

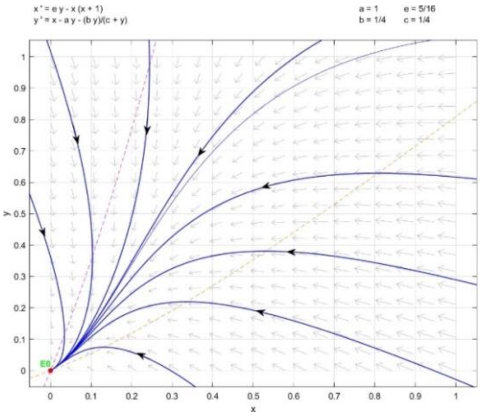


Fig. 6. There is no positive equilibrium when $b \geq c(e-a)$, $\Delta b = 0, \bar{A} > 0$.

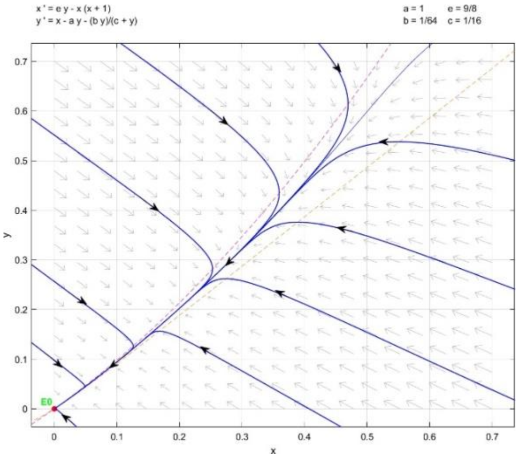


Fig. 7. There is no internal equilibrium when $c(e-a) \leq b$, $\Delta b > 0$.

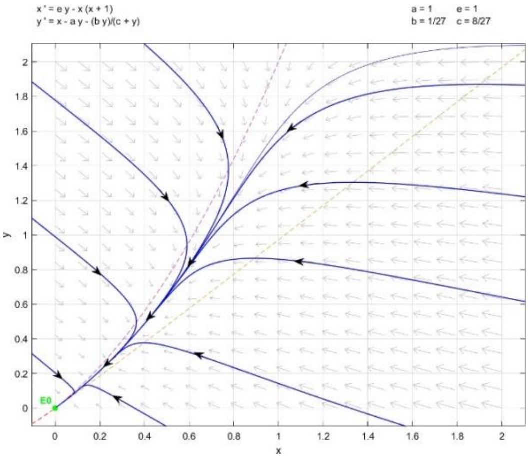


Fig. 8. There is no internal equilibrium when $c(e-a) \leq b$, $\Delta b = 0$, $\bar{A} = 0$.

On the other hand, if we take the parameter values $a = 1, b = 1, c = \frac{1}{16}, e = \frac{5}{16}$, then $\Delta b = 0$, $\bar{A} > 0$ and the system (1.8) has no internal equilibrium (figure 6).

We choose values of the parameters as $a = 1, b = \frac{1}{64}, c = \frac{1}{16}, e = \frac{9}{8}$ to get $\Delta b > 0$ for which the model has no internal equilibrium (figure 7). To verify the case when $\Delta b = 0$ and $\bar{A} = 0$, we consider the values of the parameters as $a = 1, b = \frac{1}{27}, c = \frac{8}{27}$, and $e = 1$. In this incident, the model has no internal equilibrium (figure 8). This completes the analysis.

2.5 Global Stability of Equilibria

Theorem 4. *The boundary equilibrium point $E0(0,0)$ is globally asymptotically stable.*

Proof. The function defined below is considered $V(x, y) = x + y$, as a Lyapunov function. We recall our proposed model (1.8).

$$\begin{cases} \frac{dx}{dt} = ey - x(x+1) \\ \frac{dy}{dt} = x - ay - \frac{by}{c+y} \end{cases} \quad (1.8)$$

Notice that $V(x, y)$ vanishes at $E0(0,0)$, and is positive for all $x > 0, y > 0$. We get

$$D^+V(x, y) = x' + y' = ey - x(x+1) + x - ay - \frac{by}{c+y} = -x^2 - (a-e)y - \frac{by}{c+y}$$

If $e < a$, then $D^+V(t) \leq 0$ and $D^+V(t) = 0$ iff $x = 0, y = 0$. Hence by Lyapunov's asymptotic stability theorem, $E0(0,0)$ is globally asymptotically stable[22]. The proof comes to an end here.

Theorem 5. *If $c(e-a) > b$ and $\Delta b > 0$, then the internal equilibrium point is globally asymptotically stable.*

Proof. In order to prove, we take $D(x, y) = 1$as a Dulac function. Let $P(x, y) = ey - x(x+1)$, and $Q(x, y) = x - ay - \frac{by}{c+y}$. We have

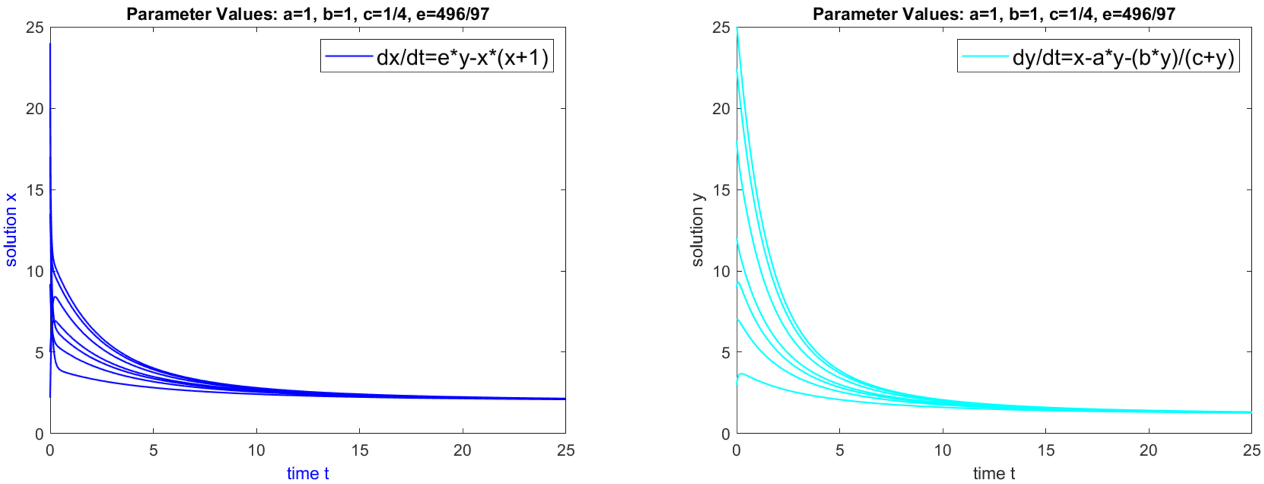
$$\frac{\partial(DP)}{\partial x} + \frac{\partial(DQ)}{\partial y} = \frac{\partial}{\partial x}\left(ey - x(x+1) + \frac{\partial}{\partial y}\left(x - ay - \frac{by}{c+y}\right)\right)$$

$$= -2x - 1 - a - \frac{(c+y)b - by}{(c+y)^2} = -2x - 1 - a - \frac{bc}{(c+y)^2}$$

The Bendixson-Dulac discriminant[22] shows that the model (1.8) can't have any limit cycle when in the first quadrant. Therefore, Hopf bifurcation is not possible[18]. Secondly, if $c(e-a) > b$, the unique boundary equilibrium $E0(0,0)$ is a saddle. As a result of the Poincare-Bendixon theorem, we can conclude that the internal equilibrium point $E1$ is globally asymptotically stable.

III. Numerical Simulation

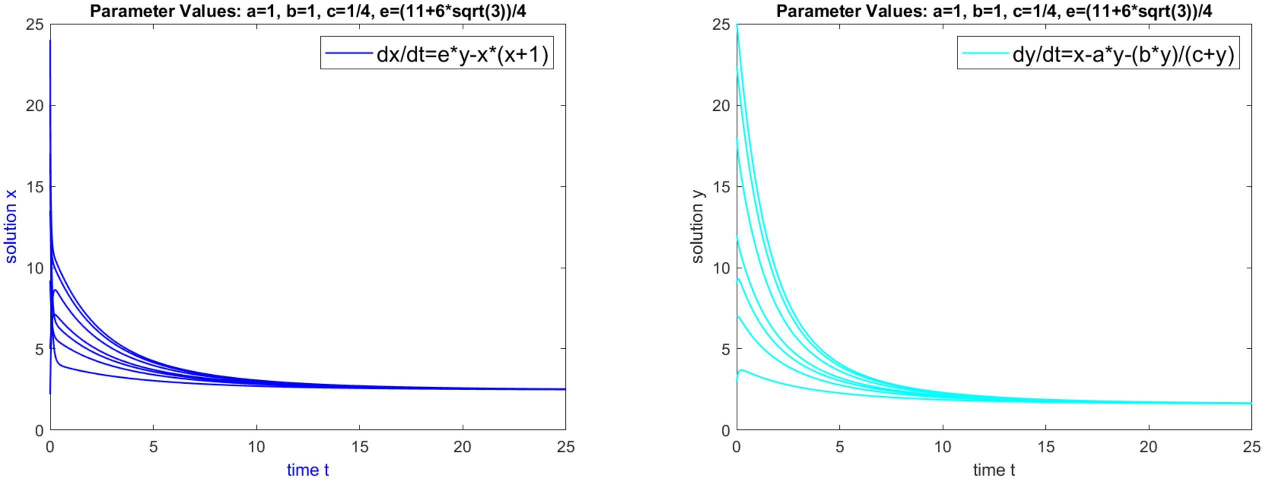
Numerical simulations of the results that we have discussed in the sections 2.3 and 2.4 are shown in the following figures.



(a) fig: Juvenile species

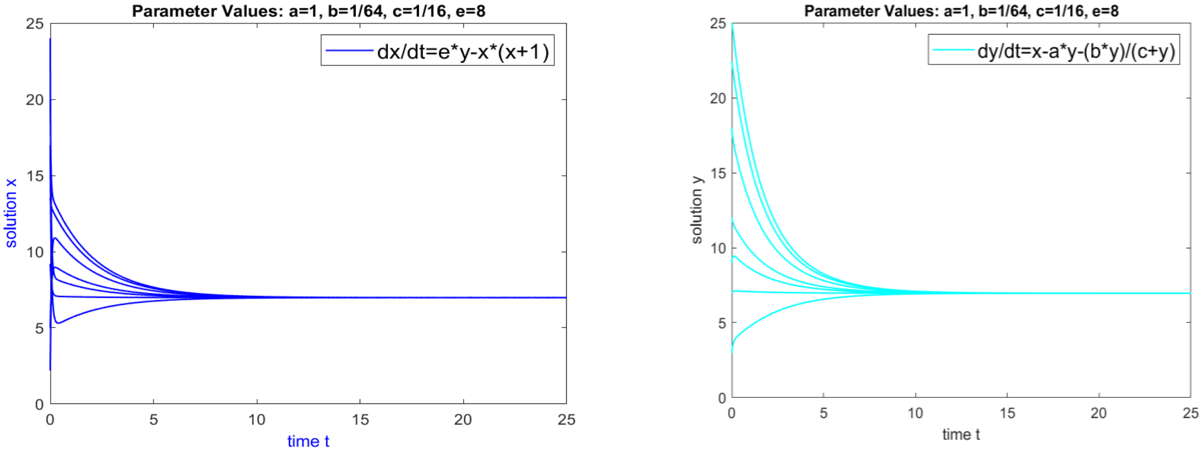
(b) fig: Adult species

Fig. 9. $\Delta b < 0$ for Case-I. **(a)** Numerical simulation of juvenile species x at the stable node $E3$. **(b)** Numerical simulation of adult species y at the stable node $E3$.



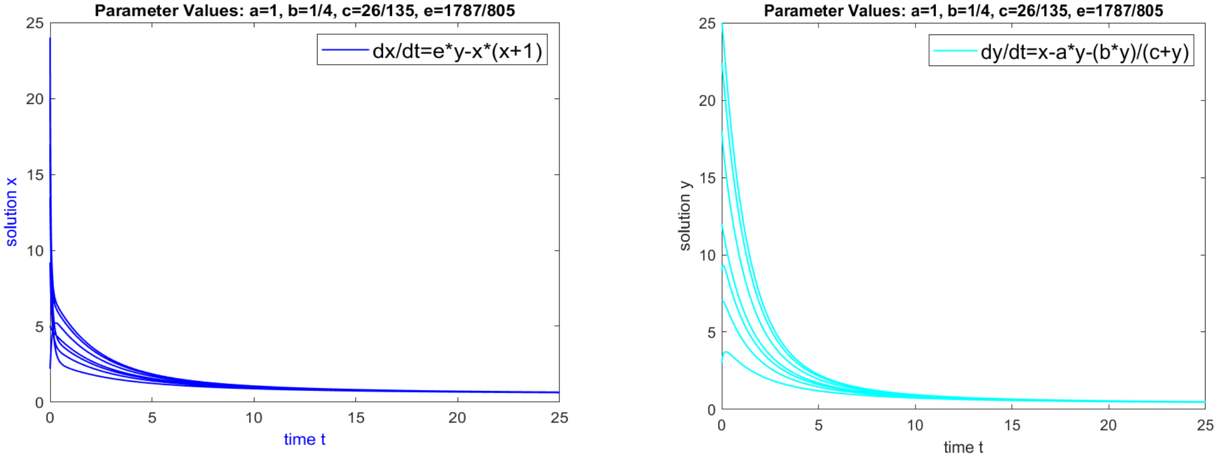
(a) fig: Juvenile species (b) fig: Adult species

Fig. 10. $\Delta b = 0$ and $\bar{A} > 0$ for Case-I. **(a)** Numerical simulation of juvenile species x at the stable node $E2$. **(b)** Numerical simulation of adult species y at the stable node $E3$.



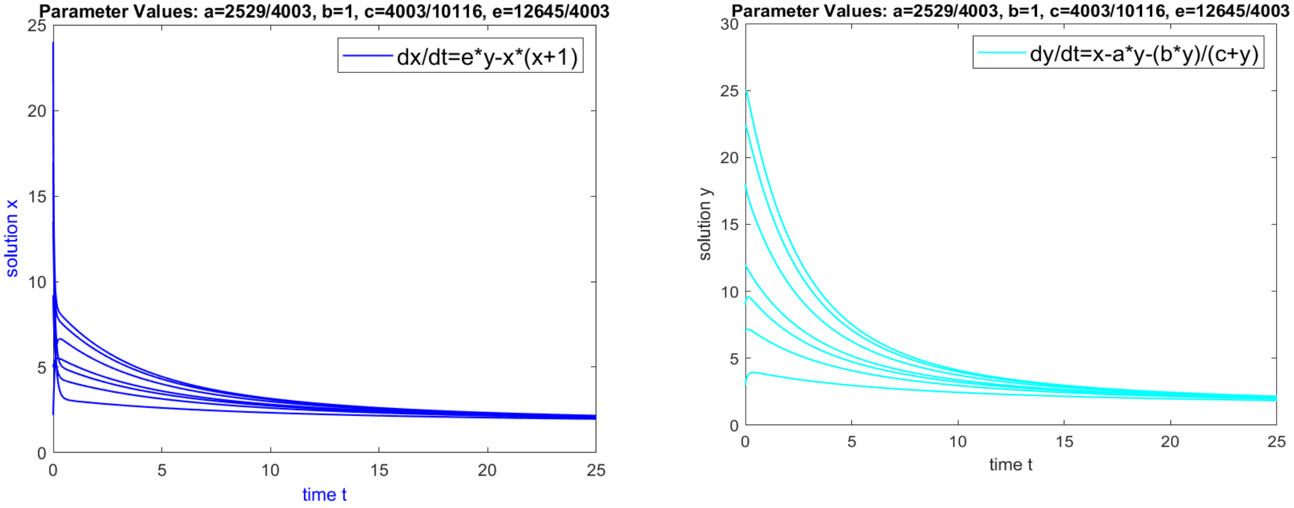
(a) fig: Juvenile species (b) fig: Adult species

Fig. 11. $\Delta b > 0$ for Case-I. **(a)** Numerical simulation of juvenile species x at the stable node $E1$. **(b)** Numerical simulation of adult species y at the stable node $E1$.



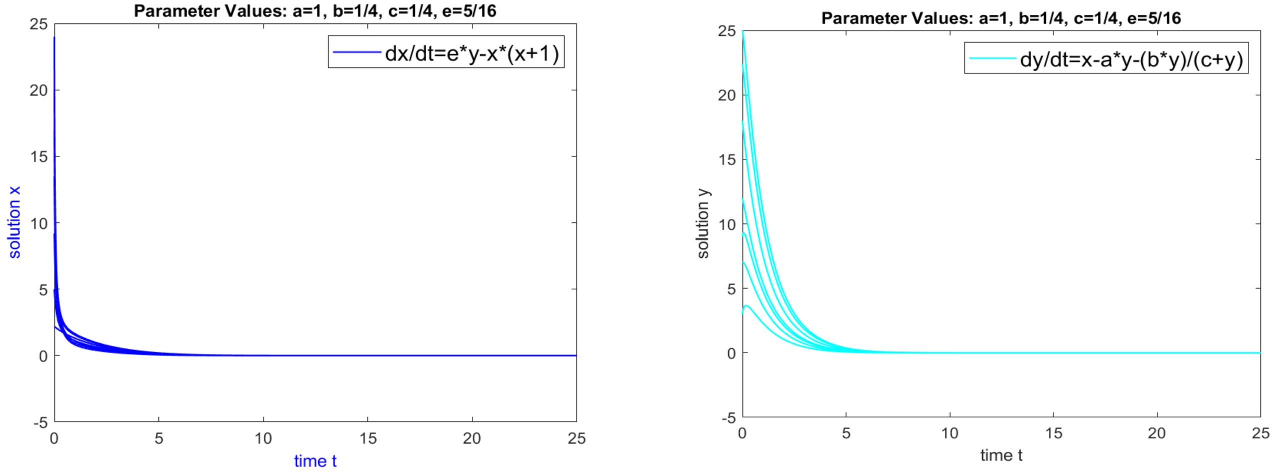
(a) fig: Juvenile species (b) fig: Adult species

Fig. 12. $\Delta b < 0$ for Case-II. **(a)** Numerical simulation of juvenile species x at the stable node $E2$. **(b)** Numerical simulation of adult species y at the stable node $E2$.



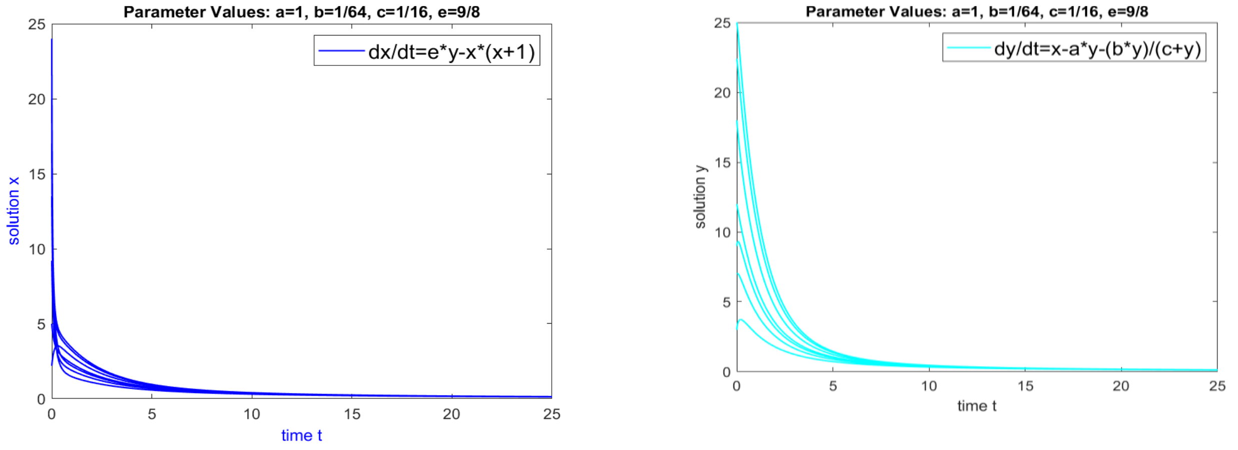
(a) fig: Juvenile species (b) fig: Adult species

Fig. 13. $\Delta b = 0$ and $\bar{A} > 0$ for Case-II. **(a)** Numerical simulation of juvenile species x. **(b)** Numerical simulation of adult species y.



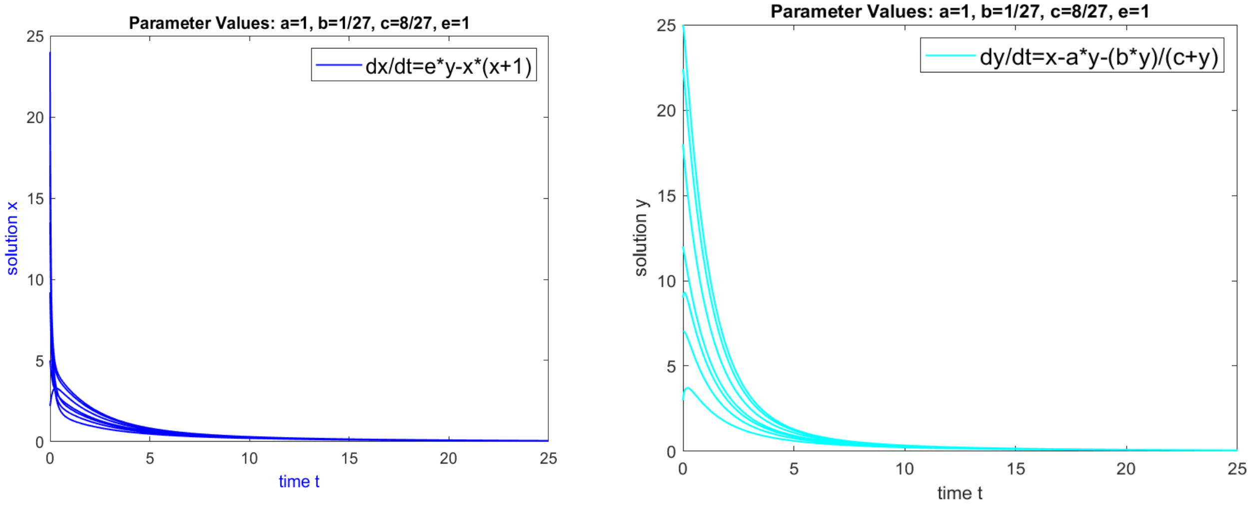
(a) fig: Juvenile species (b) fig: Adult species

Fig. 14. $\Delta b = 0$ and $\bar{A} > 0$ for Case-II. **(a)** Numerical simulation of juvenile species x at $E0$. **(b)** Numerical simulation of adult species y at $E0$.



(a) fig: Juvenile species (b) fig: Adult species

Fig.15. $\Delta b > 0$ for Case-II. **(a)** Numerical simulation of juvenile species x at $E0$. **(b)** Numerical simulation of adult species y at $E0$.



(a) fig: Juvenile species (b) fig: Adult species

Fig. 16. $\Delta b = 0$ and $\bar{A} = 0$ for Case-II. **(a)** Numerical simulation of juvenile species x at $E0$. **(b)** Numerical simulation of adult species y at $E0$.

IV. Conclusion

We have proposed and investigated a stage-structured system in which the juvenile population has to survive intra-specific competition in a single species with no harvesting strategy on the Juvenile population and a Michaelis-Menten form of harvesting on the adult population of the species. The impact of competition in the adult group with Michaelis-Menten harvesting on the adult group has been studied, reporting the coexistence and rich dynamics. Our study shows that, despite some intra-species competition in the juvenile group, we can sustainably harvest the adult species, ensuring coexistence is globally asymptotically stable. It has been observed that intra-species competition in the juvenile group may either enhance or deter the fluctuation of the total population.

V. Declaration of Interests

The authors declare no competing interests.

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