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Hopf Bifurcation of the Higher Dimensional Hénon Map

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Abstract

In this paper, the discrete time generalized Hénon map is considered and the existence of Hopf bifurcation via an explicit criterion for $N \geq 3$, in particular for $N = 4$ and $N = 5$ has given. The relation between the parameters a and b as well as the range of the values of the parameters for $N = 3, 4, 5$ has driven and the existence of Hopf bifurcation is demonstrated for the values of the parameters calculated from their relations. The results of numerical simulations for different values of the parameters are also presented.

Keywords: Hénon Map, Hopf Bifurcation.

I. Introduction

The original Hénon map is a two dimensional discrete time dynamical system[2]. The various properties of this map have been extensively studied[8,9,15,13] and the generalized form of the original map also has been studied[1,10]. The chaotic and hyperchaotic behavior for certain parameter values and initial conditions of higher dimensional generalized Hénon map have been studied by Richter[11]. Also the Hopf bifurcation of the third ordered generalized Hénon map has been studied[6]. Bifurcation means the sudden change of dynamics of a system when a parameter passes through a critical value and the type bifurcation which connects the equilibrium solutions with the periodic solution and limit cycle is called Hopf bifurcation. Routh- Hurwitz criterion gives the necessary condition for a continuous system to occur Hopf bifurcation[14]. The controlled was applied to create Hopf bifurcation in both discrete and continuous time system of arbitrary dimension[5,12]. The classical criterion of detecting the existence of Hopf bifurcation[7] is related with the eigenvalues of the Jacobian matrix of the system. Explicit Hopf bifurcation criterion was deduced from the classical criterion by Wen[4] for arbitrary dimensional maps which involve more parameters. The criterion express the relationship between the unknown parameters and the critical bifurcation constraint conditions explicitly and it is more convenient and efficient way.

In this paper, Hopf bifurcation of the higher dimensional generalized Hénon map is studied via an explicit criterion, particularly the maps with dimension higher than three are considered. This paper is organized as follows. In section 2, the generalized Hénon map is introduced. In section 3, an explicit criterion of existence of Hopf bifurcation is presented, which consists a set of simple equalities and inequalities in terms of the coefficients of the characteristic polynomial derived from the Jacobian matrix of the given system. Next in section 4, the existence of Hopf bifurcation of the third order Hénon system[6] is recalled and then the maps with dimension higher than three are considered to show the existence of Hopf bifurcation. Also the relationship between the parameters is driven in all cases in section 4. Finally in section 5, summary and conclusion are presented.

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II. The Generalized Hénon Map

The original Hénon map[2] is defined by the following pair of difference equations:

$$x_{n+1} = 1 - ax_n^2 - y_n \quad (1)$$
$$y_n = bx_n,$$

where a and b are real parameters.

The N-dimensional generalized Hénon map[1] is described by the following N-th order difference equation

$$x_1(k+1) = a - x_{N-1}^2(k) - bx_N(k) \quad (2)$$
$$x_n(k+1) = x_{n-1}(k),$$

where, $n = 2,3,\dots,N$; and $x_n, a, b \in \mathbb{R}$. For our study we consider the parameter a as a bifurcation parameter. This map contains a single quadratic term and for $N = 2$ we can obtain the original Hénon as is given in (1) by using the appropriate transformation.

There are two fixed points of the above system given by

$$x_{10}^{\pm} = x_{20}^{\pm} = \cdots = x_{N0}^{\pm} = \frac{-(1+b) \pm \sqrt{1+4a+2b+b^2}}{2}.$$

The corresponding Jacobian matrix is

$$A = \begin{bmatrix} 0 & 0 & \dots & -2x_{N-1} & -b \\ 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}$$

The determinate of A is equal to $(-1)^N b$, so the system is dissipative for $0 < |b| < 1$ [11].

The characteristic equation of the Jacobian matrix is

$$\lambda^N + \left(-1 - b \pm \sqrt{1+4a+2b+b^2}\right)\lambda$$
$$+b = 0,$$

where the coefficients are

$a_0 = 1, a_2 = a_3 = \cdots = a_{N-2} = 0, a_{N-1} = -(1+b) \pm \sqrt{1+4a+2b+b^2},\ a_N = b.$

III. An Explicit Criterion to Identify Hopf Bifurcation

The classical criterion[7] of Hopf bifurcation for maps consist of the eigenvalues assignment, the transversality condition, and the nonresonance (or resonance) condition, which are related with the properties of eigenvalues of the Jacobian matrix.

If an N-dimensional map $x_{n+1} = f_\rho(x_n)$ with fixed point x_0 satisfies the following conditions

(C1) Eigenvalue assignment: The Jacobian matrix $Df_p(x_0)$ has a pair of complex conjugate eigenvalues $\lambda_1$and $\overline{\lambda_1}$ with $|\lambda_1(\rho_0)| = 1$ and the others λ_i with $|\lambda_i(\rho_0)| < 1$, where $i = 3,4,\ldots,N$

(C2) Transversality condition: $\frac{d\lambda(\rho_0)}{d\rho} \neq 0.$

(C3) Nonresonance condition: $\lambda_1^m(\rho_0) \neq 0$

or resonance condition $\lambda_1^m(\rho_0) = 0$, $m = 3,4,5,\ldots$

then, a Hopf bifurcation occurs at $\rho = \rho_0$.

The Jacobian matrix of the higher dimensional maps may involve certain singularities, which may introduce some numeric error in eigenvalue computations. Explicit criterion expresses the relationship between the unknown parameters and the critical bifurcation constrains explicitly. It is more convenient and efficient way of detecting the existence of Hopf bifurcation for the higher dimensional maps with more parameters.

In order to express the criterion for an N-dimensional map $f_\rho$with fixed point x_0, assume that the characteristic equation has the form

$$P(\lambda) = \lambda^N + a_{N-1}\lambda^{N-1} + \cdots + a_N,$$

where $a_i = a_i(\rho, \nu)$, ρ is the bifurcation parameter and ν is the control parameter or other to be determined.

Consider the sequence of the determinants

$$\Delta_0^\pm(\rho,\nu) = 1, \Delta_1^\pm(\rho,\nu), \ldots, \Delta_N^\pm(\rho,\nu),$$

where

$$\triangle_i^\pm = \left| \begin{bmatrix} 1 & a_1 & a_2 & \ldots & a_{i-1} \\ 0 & 1 & a_1 & \ldots & a_{i-2} \\ 0 & 0 & 1 & \ldots & a_{i-3} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & 1 \end{bmatrix} \pm \begin{bmatrix} a_{N-i+1} & a_{N-i+2} & \ldots & a_{N-1} & a_N \\ a_{N-i+2} & a_{N-i+3} & \ldots & a_N & 0 \\ a_{N-i+3} & a_{N-i+4} & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_N & 0 & \ldots & 0 & 0 \end{bmatrix} \right|,$$

where $i = 1,2,3, \ldots \ldots \ldots$

Now the conditions to establish criterion of Hopf bifurcation are

(HC1) Eigenvalue assignment

$$\Delta_{N-1}^-(\rho_0,\nu) = 0, P_{\rho_0}(1) > 0, (-1)^N P_{\rho_0}(-1) > 0$$

$\Delta_N^+(\rho_0,\nu) > 0, \Delta_i^\pm(\rho_0,\nu) > 0, i = N-3, N-5, \ldots ,3,1$ (or 2) , where N is even (or odd, respectively)

(HC2) Transversality condition

$$\frac{\mathrm{d}\Delta_{N-1}^-(\rho_0,\nu)}{d\rho} \neq 0$$

(HC3) Non-resonance condition

$$cos\left(\frac{2\pi}{m}\right) \neq \psi$$

Or resonance condition

$cos\left(\frac{2\pi}{m}\right) = \psi,$

where $m = 3,4,5,\ldots$ and $\psi = 1 - 0.5\rho_0(1)\ \Delta_{N-1}^-(\rho_0,\nu)/\Delta_{N-2}^+(\rho_0,\nu)$.

If (HC1)-(HC3) hold for the map f_ρ , then Hopf bifurcation occurs for ρ_0.

IV. Hopf Bifurcation for $N \geq 3$

In this section we will show the Hopf bifurcation of the generalized Hénon map (2) for $N > 3$. But we will derive the relation between the parameters in all cases .Before that we recall the Hopf bifurcation of the map for $N = 3$ [6], then we consider the cases for $N > 3$.

The case for $N = 3$

The discrete time generalized Hénon map for $N = 3$ is described by the following third order difference equation

$$\begin{aligned} x_1(k+1) &= a - x_2^2(k) - bx_3(k) \\ x_2(k+1) &= x_1(k) \\ x_3(k+1) &= x_2(k), \end{aligned} \tag{3}$$

where x_n $(n = 1,2,3), a, b \in \mathbb{R}$ and a is the bifurcation parameter.

The fixed points of the system (2) is obtained as

$$x_{10}^\pm = x_{20}^\pm = x_{30}^\pm = \frac{-(1+b) \pm \sqrt{1+4a+2b+b^2}}{2}$$

and the corresponding Jacobian matrix is

$$A = \begin{bmatrix} 0 & -2x_{20} & -b \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

The characteristic polynomial of the Jacobian matrix

$$p(\lambda) = \lambda^3 + \left(-1 - b \pm \sqrt{1 + 4a + 2b + b^2}\right)\lambda + b$$

and the coefficients of the polynomial are

$$a_0 = 1, a_1 = 0, a_2 = -1 - b \pm\sqrt{1 + 4a + 2b + b^2}$$

Now from the condition (HC1) we have

$$\Delta_2^-(a,b) = \left|\begin{bmatrix} 1 & a_1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} a_2 & a_3 \\ a_3 & 0 \end{bmatrix}\right| = 2 + b - b^2 - \sqrt{1 + 4a + 2b + b^2} = 0$$

$$p(1) = \sqrt{1 + 4a + 2b + b^2} > 0$$

$$(-1)^3 p(-1) = -2b\sqrt{1 + 4a + 2b + b^2} > 0$$

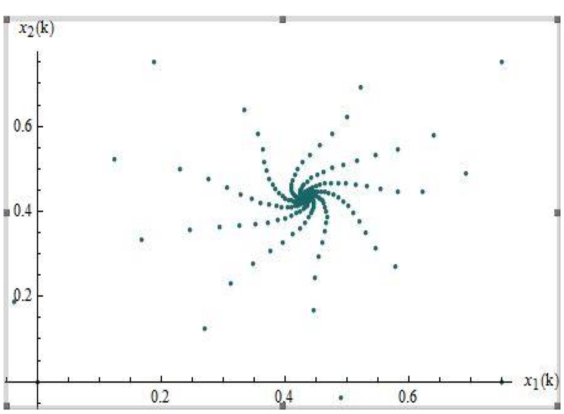
$$\Delta_2^+(a,b) = \left|\begin{bmatrix} 1 & a_1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a_2 & a_3 \\ a_3 & 0 \end{bmatrix}\right| = -b - b^2 + \sqrt{1 + 4a + 2b + b^2} > 0$$

Solving the above equality and inequalities, we obtain a relation between a and b as

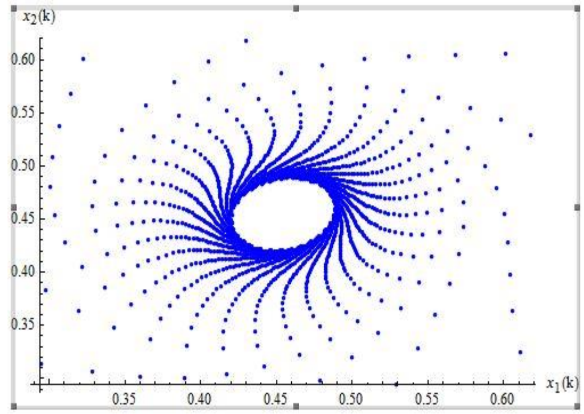
$$-1 < b < 1, a = \frac{3 + 2b - 4b^2 - 2b^3 + b^4}{4}$$

And the range of the values of a and b are $-1 < b < 1,\ 0 < a < 0.806862$. Therefore, if we chose any value of b in the interval $(-1, 1)$, then we can find a value of a for which the map (3) shows Hopf bifurcation. For example, let $b = 0.5$ then $a = 0.7031125$ and the equilibrium point is given by $(x_{10}, x_{20}, x_{30}) = (0.375, 0.375, 0.375)$. The eigenvalues of the Jacobian matrix are $\lambda_1 = -0.5, \lambda_{2,3} = 0.25 \pm 0.968246i$ and $|\lambda_1| = 0.5$, $|\lambda_{2,3}| = 1$, which satisfy the first condition of Hopf bifurcation. Therefore, Hopf bifurcation occurs at $(x_{10}, x_{20}, x_{30}) = (0.375, 0.375, 0.375)$ for

$$b = 0.5 \text{ and } a = 0.703125.$$



(a) 1(a)



(b) 1(b)

Fig. 1. Hopf bifurcation attractor for $N = 3$. Figure (1a) shows the attracting fixed point before bifurcation when $a = 0.75, b = 0.30$ and figure (1b) shows the limit cycle for $a = 0.703125, b = 0.30$.

The case for N = 4

Now we consider the case for $N = 4$. The generalized Hénon map for $N = 4$ can be expressed by the following fourth order difference equations

$$\begin{aligned} x_1(k+1) &= a - x_3^2(k) - bx_4(k) \\ x_2(k+1) &= x_1(k) \\ x_3(k+1) &= x_2(k) \\ x_4(k+1) &= x_3(k) \end{aligned} \qquad (4)$$

where $x_n(n = 1, 2, 3, 4)$; a; $b \in \mathbb{R}$ and we consider a as the bifurcation parameter.

The fixed points of the system (4) are given by

$$x_{10}^{\pm} = x_{20}^{\pm} = x_{30}^{\pm} = x_{40}^{\pm} = \frac{-(1+b) \pm \sqrt{1 + 4a + 2b + b^2}}{2},$$

and the corresponding Jacobian matrix of the system is

$$A = \begin{bmatrix} 0 & 0 & -2x_{30} & -b \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The characteristic polynomial is

$$P(\lambda) = \lambda^4 + (-1 - b \pm \sqrt{1 + 4a + 2b + b^2})\lambda + b = 0,$$

where the coefficients are $a_0 = 1, a_1 = a_2 = 0, a_3 = -1 - b \pm \sqrt{1 + 4a + 2b + b^2}, a_4 = b.$

Now from the firrst condition of the explicit criterion, we obtain the following set of equality and inequalities

$$\Delta_3^-(a,b) = \left|\begin{bmatrix} 1 & a_1 & a_2 \\ 0 & 1 & a_1 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} a_2 & a_3 & a_4 \\ a_3 & a_4 & 0 \\ a_4 & 0 & 0 \end{bmatrix}\right| = -1 - 4a - 5b - 3b^2 + b^3 + 2\sqrt{1 + 4a + 2b + b^2} + 2b\sqrt{1 + 4a + 2b + b^2} = 0$$

$$P(1) = \sqrt{1+4a+2b+b^2} > 0$$

$$(-1)^4 P(-1) = 2 + 2b\sqrt{1+4a+2b+b^2} > 0$$

$$\Delta_3^{+}(a,b) = \left|\begin{bmatrix}1 & a_1 & a_2\\ 0 & 1 & a_1\\ 0 & 0 & 1\end{bmatrix} + \begin{bmatrix}a_2 & a_3 & a_4\\ a_3 & a_4 & 0\\ a_4 & 0 & 0\end{bmatrix}\right| = -1-4a-3b-3b^2-b^3 + 2\sqrt{1+4a+2b+b^2} + 2b\sqrt{1+4a+2b+b^2} > 0$$

and $\Delta_1^{\pm}(a,b) = 1 \pm \mathrm{b} > 0$.

Solving the above equality and inequalities, we have the following relation between a and b as

$0 < b < 1$and

$$a = \frac{1-b-b^2+b^3-2\sqrt{1+b-2b^2-2b^3+b^4+b^5}}{4}$$

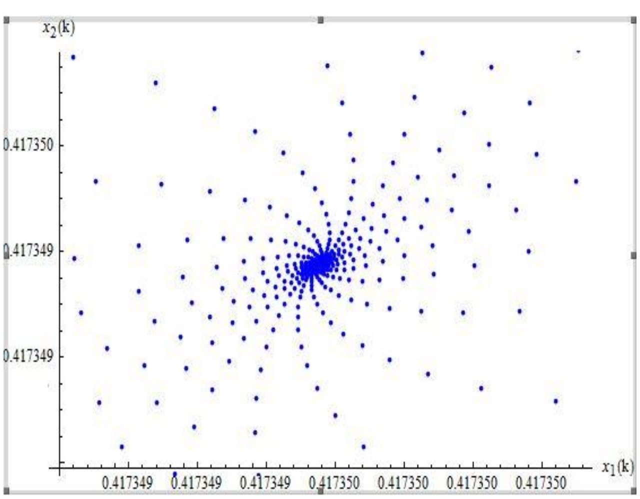
or $\mathrm{a} = \frac{1-b-b^2+b^3+2\sqrt{1+b-2b^2-2b^3+b^4+b^5}}{4}$

and the ranges of the values of a and b is $0 < b < 1, -\frac{1}{4} < a < 0$ and $0 < a < \frac{3}{4}$.

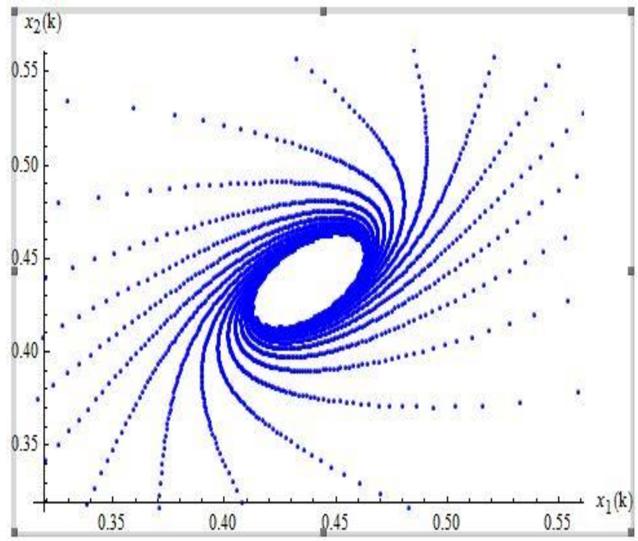
So for any paired values of a and b obtained from the above relation, Hopf bifurcation of the map (4) occurs for those values. In particular, let $b = 0.2$ then $a = 0.71781$, so the equilibrium point is given by $(x_{10}, x_{20}, x_{30}, x_{40}) = (0.438176, 0.438176, 0.438176, 0.438176)$ and the eigenvalues of the Jacobian matrix for this equilibrium point are $\lambda_1 = -0.863947, \lambda_2 = -0.231496, \lambda_{3,4} = 0.547722 \pm 0.836658i$ and $|\lambda_1| = 0.863947, |\lambda_2| = 0.231496, |\lambda_{3,4}|$

$= 1$, which satisfy the condition (HC1) for existence of Hopf bifurcation. Therefore, Hopf bifurcation occurs at $(x_{10}, x_{20}, x_{30}, x_{40}) = (0.438176, 0.438176, 0.438176, 0.438176)$for

$$b = 0{:}2 \text{ and } a = 0{:}71781.$$



(a) 2(a)



(b) 2(b)

Fig. 2. Hopf bifurcation attractor for $N = 4$. Figure 2(a) shows the attracting fixed point before bifurcation when $a = 0.67, b = 0.20$ and figure 2(b) shows the limit cycle for $a = 0.71781, b = 0.20$.

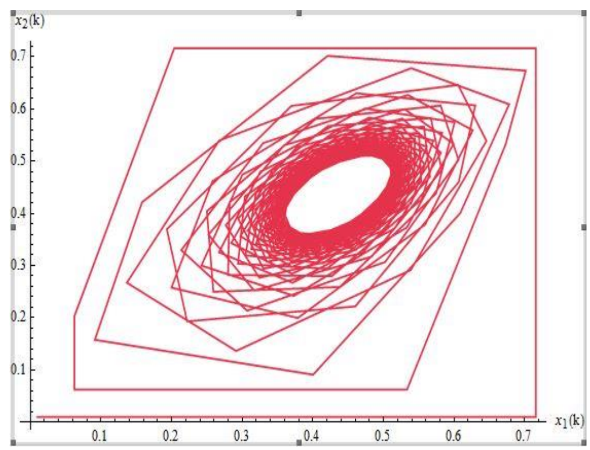


Fig. 3. Phase plot of the map for $N = 4$.

The case for N = 5

The generalized Hénon map for $N = 5$ is given by the following difference equation

$$\begin{aligned} x_1(k+1) &= a - x_4^2(k) - bx_5(k)\\ x_2(k+1) &= x_1(k)\\ x_3(k+1) &= x_2(k)\\ x_4(k+1) &= x_3(k)\\ x_5(k+1) &= x_4(k)\end{aligned} \qquad (5)$$

The fixed points of the system are given by =

$$x_{10}^{\pm} = x_{20}^{\pm} = x_{30}^{\pm} = x_{40}^{\pm} = x_{50}^{\pm} = \frac{-(1+b) \pm \sqrt{1+4a+2b+b^2}}{2}$$

Jacobian matrix of the system at fixed point is

$$A=\begin{bmatrix} 0 & 0 & 0 & -2x_{40} & -b \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

The characteristic polynomial is

$$\lambda^5 + \left(-1 - b \pm \sqrt{1+4a+2b+b^2}\right)\lambda$$
$$+b = 0$$

and the coefficients are

$$a_0 = 1,\ a_1 = 0,\ a_2 = 0,\ a_3 = 0,$$
$$a_4 = -1 - b \pm \sqrt{1+4a+2b+b^2}\,, a_5 = b.$$

Now we have

$$\Delta_4^-(a,b) = \left|\begin{bmatrix} 1 & a_1 & a_2 & a_3 \\ 0 & 1 & a_1 & a_2 \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} a_2 & a_3 & a_4 & a_5 \\ a_3 & a_3 & a_5 & 0 \\ a_4 & a_5 & 0 & 0 \\ a_5 & 0 & 0 & 0 \end{bmatrix}\right|$$
$$= -4 - 16a - 15b - 12ab - 17b^2 - 5b^3 + b^4 + 5\sqrt{1+4a+2b+b^2} + 4a\sqrt{1+4a+2b+b^2} + 10b\sqrt{1+4a+2b+b^2} + 5b^2\sqrt{1+4a+2b+b^2}$$

$$P(1) = \sqrt{1+4a+2b+b^2} > 0$$

$$(-1)^5P(-1) = -2b + \sqrt{1+4a+2b+b^2} > 0$$

$$\Delta_4^+(a,b) = \left|\begin{bmatrix} 1 & a_1 & a_2 & a_3 \\ 0 & 1 & a_1 & a_2 \\ 0 & 0 & 1 & a_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} a_2 & a_3 & a_4 & a_5 \\ a_3 & a_3 & a_5 & 0 \\ a_4 & a_5 & 0 & 0 \\ a_5 & 0 & 0 & 0 \end{bmatrix}\right|$$
$$= 2 + 8a + 7b + 12ab + 9b^2 + 5b^3 + b^4 - \sqrt{1+4a+2b+b^2} - 4a\sqrt{1+4a+2b+b^2} - 6b\sqrt{1+4a+2b+b^2} - 5b^2\sqrt{1+4a+2b+b^2} > 0$$

$$\Delta_2^-(a,b) = \left|\begin{bmatrix} 1 & a_1 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} a_4 & a_5 \\ a_5 & 0 \end{bmatrix}\right|$$
$$= 2 + b - b^2 - \sqrt{1+4a+2b+b^2} > 0$$

$$\Delta_2^+(a,b) = \left|\begin{bmatrix} 1 & a_1 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a_4 & a_5 \\ a_5 & 0 \end{bmatrix}\right|$$
$$= -b - b^2 - \sqrt{1+4a+2b+b^2} > 0$$

Solving the equality and inequalities, we obtain the following relations between a and b :

$-1 < b < 0$ *or* $0 < b < 1$ and a is the second root of the equation

$$64x^3 + (80 - 32b + 32b^2)x^2 + (12 - 16b - 64b^2 - 8b^3 + 52b^4 + 24b^5)x + 9 + 30b + 14$$

$$64x^3 + (80 - 32b + 32b^2)x^2 + (12 - 16b - 64b^2 - 8b^3 + 52b^4 + 24b^5)x + 9 + 30b + 14b^2 - 50b^3 - 56b^4 + 10b^5 + 34b^6 + 10b^7 - b^8 = 0$$

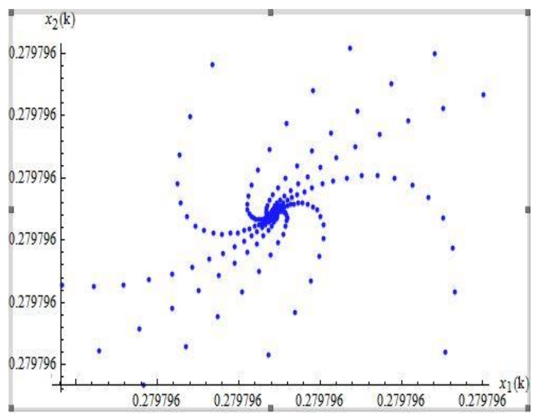
From this relation we see that the ranges of values of the parameters are $-1 < b < 0$ or $0 < b < 1$ and $0 < a < \frac{3}{4}$. In particular, let $b = 0.4$ then $a = 0.57465$, so the equilibrium point is given by $(x_{10}, x_{20}, x_{30}, x_{40}\ x_{50}) = (0.33182, 0.33182, 0.33182, 0.33182, 0.33182)$, and the eigenvalues of the Jacobian matrix for this equilibrium point are $\lambda_1 = -0.536045$; $\lambda_{2,3} = 0.484596 \pm 0.715105i$;

$\lambda_{4,5} = 0.752618 \pm 0.658456i$ and $|\lambda_1| =$

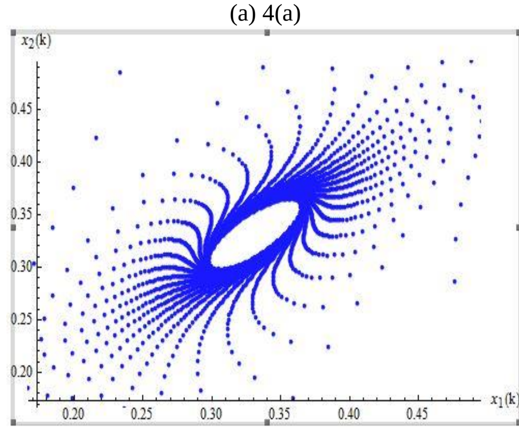
0.536045; $|\lambda_{2,3}| = 0.863833$; $|\lambda_{4,5}| = 1$,

which satisfy the condition of existence of Hopf bifurcation. Therefore, Hopf bifurcation occurs at $x_{10} = x_{20} = x_{30} = x_{40} = x_{50} = 0.33182$

for $b = 0.4$ and $a = 0.57465$.



(a) 4(a)



(b) 4(b)

Fig. 4. Hopf bifurcation attractor for $N = 5$. Figure (4a) shows the attracting fixed point before bifurcation when $a = 0.47, b = 0.40$ and figure (4b) shows the limit cycle for $a = 0.57465, b = 0.40$.

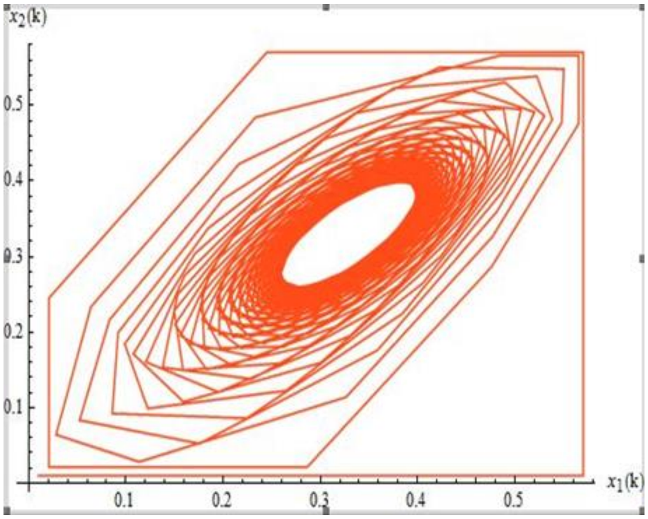


Fig. 5. Phase plot of the map for $N = 5$.

V. Conclusion

In the paper, we have considered the generalized Hénon map and then applied an explicit criterion of Hopf bifurcation to the map. The relationship between the parameters a and b for $N = 3$ has been derived. Also the existence of Hopf bifurcation has been shown for $N = 4$, and $N = 5$. We also derived the relation between the parameters a and b for $N = 4, 5$ and ranges of the values of the parameters. The mentioned results have been shown via numerical simulations representing them in the phase space.

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