

# A Comparative Analysis of Monte Carlo and Quasi-Monte Carlo Methods in Financial Derivative Pricing

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## Abstract

The main purpose of this dissertation is to study Monte Carlo (MC) and Quasi-Monte Carlo (QMC) methods for pricing financial derivatives. We estimate the Price of European as well as various path dependent options like Asian, Barrier and American options by using these methods. We also compute the numerical results by the above mentioned methods and compare them graphically as well with the help of the **MATLAB** Coding.

**Keywords:** Monte Carlo (MC) method, Quasi-Monte Carlo (QMC) method, European option, American option, Asian option, Barrier option, Pseudorandom number, Halton sequence, Sobol sequence, Black Scholes model.

## I. Introduction

The Monte Carlo (MC) method and its deterministic version Quasi-Monte Carlo (QMC) method are popular numerical tools in pricing financial derivatives and risk management. Monte-Carlo method and Quasi Monte-Carlo method were both used in physics at the beginning, so most solutions were given by physicists. There had been a controversy about whether these solutions also applicable in finance and derivatives pricing. Corwin Joy<sup>8</sup> (Corwin, 1996) pointed out the disadvantage of Monte-Carlo and raised Quasi Monte-Carlo. They substituted random sequence with deterministic sequence which can improve convergent rate and have a deterministic error interval.<sup>5</sup>

The MC method simulates the underlying model using random numbers, and estimates the expectation of the random variable by sample averages. The QMC method attempts to do the estimation similar to the MC method with one crucial difference that it uses the so-called low-discrepancy sequences in simulation. These are specially designed number theoretic sequences for estimating integrals. For this reason, the QMC method is often called the deterministic version of the MC method. There are two advantages of using the QMC method: it provides deterministic convergence, as opposed to the stochastic convergence of MC method. Moreover, the QMC estimates converge to the true solution at the rate  $O((\log N)^d/N)$ , where  $N$  is sample size and  $d$  is the dimension of the problem whereas the MC convergence rate is  $O(N^{-1/2})$ . The QMC method has some drawbacks: first, using one low discrepancy sequence in simulation gives a single estimate for the integral, and there is no practical way of assessing the error of this estimate. The second drawback is due to the deterioration of low discrepancy sequences in high dimensions. The third main shortcoming is due to the lack of practical bounds for integration error in QMC methods. Therefore, more practical error bounds are needed in QMC integration.<sup>5</sup> Geometric Brownian motion is a standard model used in financial mathematics, and the MC method is often used to simulate this model. Generation of these paths requires the simulation of independent standard normal random

variables at each time increment. Therefore, we need a good method to generate these standard normal random variables. We use two well known methods; Inverse transformation and the Box-Muller methods for generating the standard normal random variables.<sup>3</sup>

## II. The Monte Carlo Method for Option Pricing

Consider the problem of estimating an integral over the unit interval. We can represent it in the following form<sup>1</sup>

$$I = \int_0^1 f(x)dx = \int_0^1 f(x)p(x)dx \quad (1)$$

where,  $p$  is the uniform density. The basic Monte Carlo estimator is

$$I_N = \frac{1}{N} \sum_{k=1}^N f(x_k) \quad (2)$$

and the Monte Carlo error can be approximated by the sample standard deviation of<sup>5</sup>

$$\sigma_f = E[f - E[f]]^2]^{1/2}$$

We assume that the underlying asset follows a Geometric Brownian motion where in a risk-neutral world, the drift term is equal to the risk free interest rate. In a continuous time notation, the stock price is:

$$dS = rSdt + \sigma Sdz \quad (3)$$

Where  $S$  is the stock price,  $r$  is the risk-free interest rate,  $\sigma$  is the volatility which we assumed to be a constant, and  $dz$  is a Wiener process. Here,  $\Delta z = \varepsilon \Delta t$  and  $\varepsilon$  follows the standard normal distribution with mean zero and variance of one. Therefore, for a discrete time system, a change in stock price becomes

$$\begin{aligned} \Delta S &= rS\Delta t + \sigma S\Delta z \\ &= rS\Delta t + \sigma S\varepsilon\sqrt{\Delta t} \end{aligned} \quad (4)$$

$$\text{Thus } \frac{\Delta S}{S} = r\Delta t + \sigma\varepsilon\sqrt{\Delta t} \quad (5)$$

which also follows a normal distribution, and represents a percentage change in stock return over a short time period  $\Delta t$ . Now we follow the procedure for Monte Carlo simulation and divide the lifespan of the stock into  $n$  short

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intervals with length of  $\Delta t$ , then by applying Ito's Lemma to the stock price process, we get:

$$\begin{aligned} d \ln(S) &= \left( \frac{1}{S} rS + 0 + \frac{1}{2} \left( -\frac{1}{S^2} \right) \sigma^2 S^2 \right) dt + \frac{1}{S} \sigma S dz \\ &= \left( r - \frac{1}{2} \sigma^2 \right) dt + \sigma dz \end{aligned} \quad (6)$$

By applying to discrete time notation, the above formula becomes:

$$d \ln(S) = \left( r - \frac{1}{2} \sigma^2 \right) \Delta t + \sigma \varepsilon \sqrt{\Delta t} \quad (7)$$

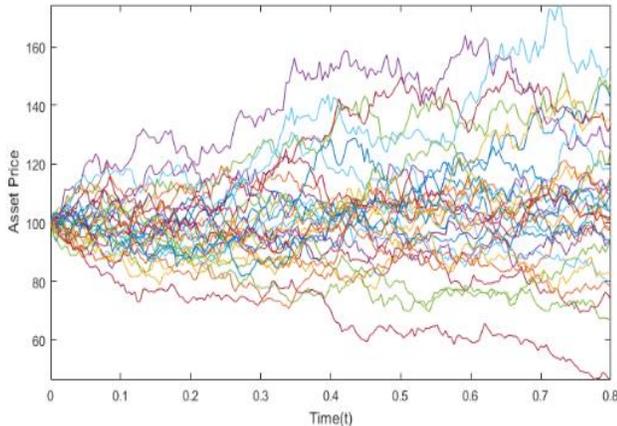
Since  $\Delta t$  represents short time interval, the change in stock price becomes:

$$\begin{aligned} \ln(S_{t+\Delta t}) - \ln(S_t) &= \left( r - \frac{1}{2} \sigma^2 \right) \Delta t + \sigma \varepsilon \sqrt{\Delta t} \end{aligned}$$

Which eventually gives the path generating formula for the stock price by applying Monte Carlo simulation

$$S_{t+\Delta t} = S_t e^{\left( r - \frac{1}{2} \sigma^2 \right) \Delta t + \sigma \varepsilon \sqrt{\Delta t}} \quad (8)$$

Where,  $S_t$  denotes the value of the stock at time  $t$ ,  $\varepsilon$  represents a number randomly sampled from a normal distribution with zero mean and standard deviation equals to one and when the volatility is constant, the parameters  $\mu$  and  $\sigma$  are also constant, which makes the above equation the true value of stocks, instead of an approximation following Geometric Brownian motion, which is then used to value the price of an European as well as various path dependent options by using Monte Carlo simulation. For this paper, the formula is encoded into **MATLAB** to create a series of random paths (Fig. 1) following Brownian motion.<sup>4</sup>

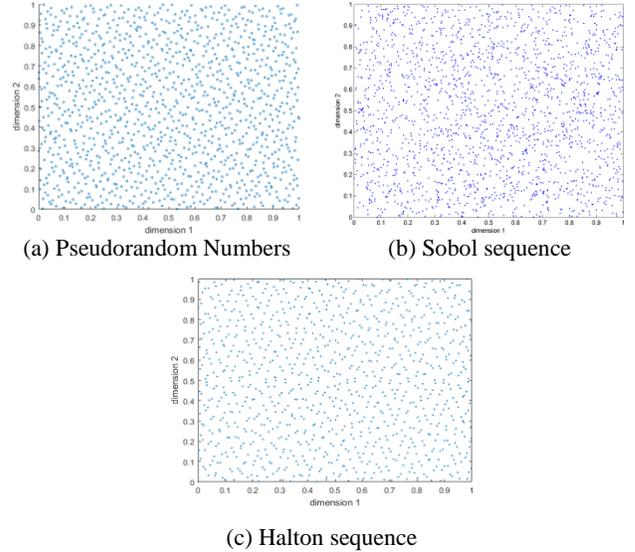


**Fig. 1.** Standard Construction of Brownian motion path

### III. Pseudorandom vs Quasi-Random Points

The value of Monte Carlo integral highly depends on the choice of random number. The difference between quasi-Monte Carlo and Monte Carlo is the way the random numbers are chosen. Quasi-Monte Carlo uses a low-discrepancy sequence such as the Halton sequence,

the Sobol sequence, whereas Monte Carlo uses a pseudorandom sequence. The advantage of using low-discrepancy sequences is a faster rate of convergence. From Fig. 2, we see that Halton and Sobol sequence is more evenly distributed compared to Pseudorandom number.



**Fig. 2.** Comparison: Random points vs Quasi-Random points (Sobol and Halton sequence).

### IV. The Quasi-Monte Carlo Method

Consider the following  $d$  dimensional integral,

$$\int_{[0,1]^d} f(x) dx$$

The QMC method approximates this integral with

$$\int_{[0,1]^d} f(x) dx \approx \sum_{i=1}^N f(u_i)$$

where,  $u_1, \dots, u_N$  are coming from a low discrepancy sequence instead of a pseudorandom numbers.

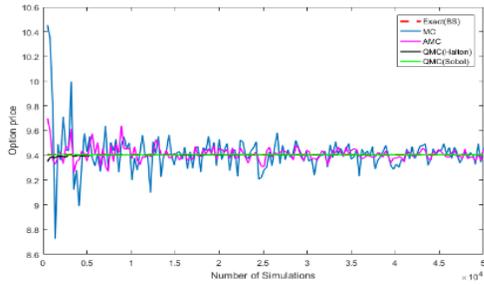
### V. MC and QMC Methods for European Call Option Pricing

We consider an European Call options with initial stock price  $S_0 = 90$ ,  $\sigma = 0.2$ ;  $r = 0.05$ ;  $n = 10,000$ . By varying  $K$  and  $T$ , we get Table 1 which shows that European call option price increases when time to maturity increases. For MC method, Antithetic variate gives better results than standard MC. For QMC method, both Halton and Sobol sequence gives the option price which is very close to BS option price.

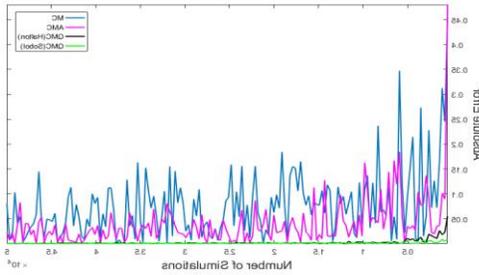
Fig. 3(a) shows the European call option price with MC, AMC and QMC method by using Halton and Sobol sequence. From this figure, we see that QMC method converges much more faster than MC method.

**Table 1. European Call option pricing by MC and QMC Methods with  $S_0 = 90$ ,  $\sigma = 0.2$ ,  $r = 0.05$ ,  $n = 10,000$  for different  $K$  and  $T$ .**

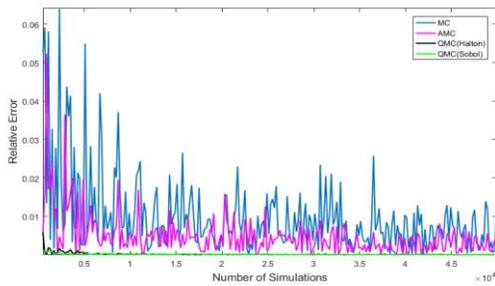
K	T	Exact	Monte Carlo		Quasi-Monte Carlo	
			Standard	AMC	Halton	Sobol
90	1	9.4055	9.1652	9.4655	9.4053	9.4048
	2	14.514	14.4594	14.5688	14.5138	14.5126
	3	18.8319	18.7706	18.7753	18.8317	18.8295
	4	22.6919	22.8713	22.9269	22.6920	22.6884
	5	26.2247	26.1451	26.1033	26.2250	26.2200
100	1	5.0912	5.3314	5.2079	5.0917	5.0906
	2	9.9088	10.0278	9.9639	9.9091	9.9074
	3	14.1690	14.1451	14.2444	14.1698	14.1674
	4	18.0602	17.8771	18.0831	18.0606	18.0569
	5	21.6677	21.6358	21.5537	21.6682	21.6632
110	1	2.5237	2.4919	2.5063	2.5243	2.5231
	2	6.5545	6.7856	6.5754	6.5554	6.5531
	3	10.4927	10.3730	10.6026	10.4937	10.4908
	4	14.2413	14.5854	14.2618	14.2423	14.2383
	5	17.7998	17.8581	17.8420	17.8008	17.7954



(a) Comparison of MC and QMC methods



(b) Absolute error



(c) Relative error

**Fig. 3.** Comparison and error of MC and QMC methods for European Call option price.

For higher number of simulations, both MC and QMC method converges to the BS option price. Fig. 3(b) shows

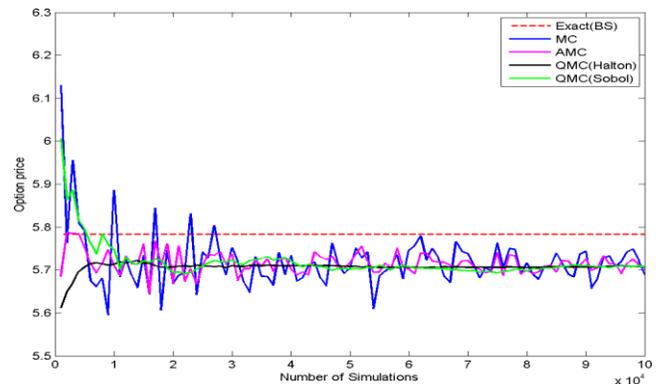
the absolute error of MC and QMC method for European Call option pricing. As  $n$  increases, the absolute error of both MC and QMC method decreases. Here, QMC method gives the lower relative error than MC method. From Fig. 3(c), we see that relative error of QMC method is very small compared with MC method.

**VI. Comparison Between MC and QMC Methods for Asian Option Pricing**

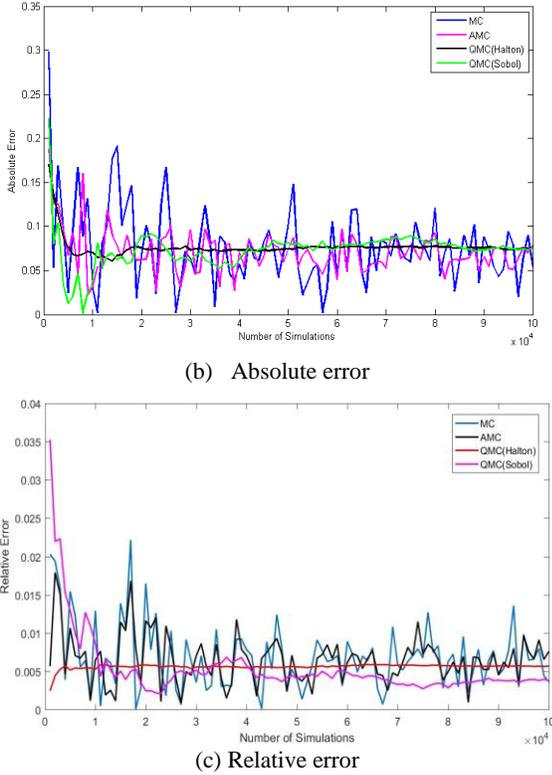
The following Table 2 shows the arithmetic average Asian call option price with Standard MC, AMC and QMC method by using Halton and Sobol sequence by using the parameter  $S_0 = 100$ ;  $r = 0.05$ ;  $\sigma = 0.2$ ;  $T = 1$  and by varying  $K$  and  $m$  (Number of Steps). Here, we consider extended BS formula for Asian option pricing as standard.<sup>2</sup> When number of steps is higher, QMC method with Halton sequence gives worse result whereas QMC method with Sobol sequence gives better result for any number of steps.

**Table 2. Asian option pricing by MC and QMC method with  $S_0 = 100$ ,  $r = 0.05$ ,  $\sigma = 0.2$ ,  $T = 1$ ,  $n = 1,00,000$  for different  $K$  and  $m$ .**

K	m	Exact (BS)	MC		QMC	
			Standard	AMC	Halton	Sobol
90	10	12.6298	12.5835	12.5470	12.545	12.5393
	20	12.6298	12.5766	12.5679	12.5684	12.5560
	50	12.6298	12.5812	12.5982	14.211	12.5324
	100	12.6298	12.6173	12.5921	14.312	12.542
	200	12.6298	12.5877	12.5986	14.0036	12.5534
	500	12.6298	12.5920	12.5969	13.5334	12.5990
100	10	5.7828	5.7038	5.6799	5.6628	5.6668
	20	5.7828	5.7232	5.7347	5.7076	5.7073
	50	5.7828	5.7464	5.7407	6.8855	5.7155
	100	5.7828	5.7540	5.7625	6.9535	5.7192
	200	5.7828	5.7630	5.7670	6.7308	5.7166
	500	5.7828	5.7540	5.7594	6.3628	5.7493
110	10	1.9711	1.9249	1.9262	1.9154	1.9215
	20	1.9711	1.9310	1.9308	1.9523	1.9539
	50	1.9711	1.9843	1.9794	2.5484	1.9542
	100	1.9711	1.9649	1.9834	2.5660	1.9608
	200	1.9723	1.9969	1.9820	2.4454	1.9612
	500	1.9114	1.9915	1.9932	2.2181	1.9769



(a) Comparison of MC and QMC methods



**Fig. 4.** Comparison and error of MC and QMC methods for arithmetic Asian call option price.

Fig. 4(a) shows the arithmetic average Asian call option price by using MC, AMC and QMC methods. For higher number of simulation, both MC and QMC methods gives similar result but QMC method converges faster than MC method. Fig. 4(b) and 4(c) shows the absolute error and relative error of MC and QMC methods for arithmetic average Asian call option. Here, QMC methods gives lower absolute as well as relative error compared to MC methods.

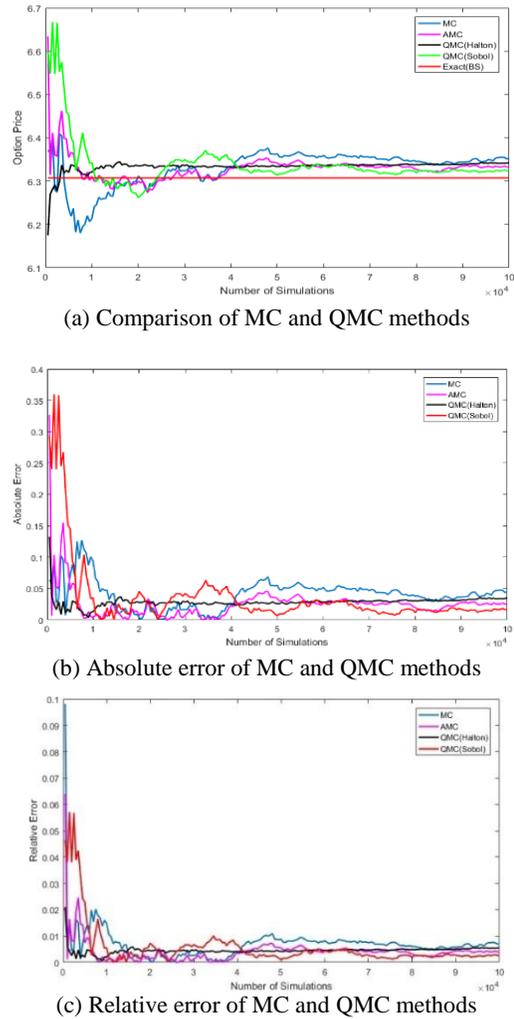
**VII. Barrier Option Pricing by MC and QMC Methods**

The following Table 3 shows the price of a Down-and-Out Barrier Call option with MC and QMC methods with  $K = 100$ ,  $r = 0.1$ ,  $\sigma = 0.3$ ,  $T = 0.2$  and by varying Initial Stock price  $S_0$  and Barrier level  $B$  by considering 100,000 random paths. Here, we consider extended BS formula for barrier option pricing as standard.<sup>7</sup> From Table 3, we see that QMC methods (with Halton and Sobol sequence) gives better results compared to MC and AMC methods.

Fig. 5(a) shows the convergence of MC and QMC methods towards Extended BS for Down-and-Out Barrier Call option pricing. As number of simulations increases, both MC and QMC method gives option price which is very close to Extended BS price for Barrier option. The QMC method performed extremely well, as convergence is achieved faster and this was in connection to the use of Sobol sequences.

**Table 3. Down and Out Barrier Call option pricing by MC and QMC methods with  $K = 100$ ,  $r = 0.1$ ,  $\sigma = 0.3$ ,  $T = 0.2$  and varying  $S_0$  and  $B$ .**

$S_0$	B	Exact (BS)	MC		QMC	
			Standard	AMC	Halton	Sobol
100	80	6.3425	6.3524	6.3325	6.3420	6.3232
	85	6.3076	6.3364	6.3169	6.3273	6.3084
	90	5.9772	6.1583	6.1399	6.1485	6.1339
	95	4.3975	5.1613	5.1411	5.1373	5.1206
105	80	9.6158	9.6275	9.5974	9.6104	9.5854
	85	9.6052	9.6234	9.5934	9.6060	9.5808
	90	9.4733	9.5581	9.5283	9.5423	9.5176
	95	8.6732	9.0875	9.0603	9.0681	9.0445
110	80	13.484	13.5006	13.4616	13.475	13.4442
	85	13.481	13.4996	13.4603	13.473	13.4433
	90	13.431	13.4768	13.4387	13.452	13.4185
	95	13.051	13.2736	13.2315	13.247	13.2195
115	80	17.793	17.8113	17.7680	17.787	17.7434
	85	17.792	17.8110	17.7677	17.787	17.7431
	90	17.774	17.8031	17.7599	17.778	17.7356
	95	17.604	17.715	17.6739	17.695	17.6451



**Fig. 5.** Comparison, absolute and relative error of MC and QMC methods for Barrier option price.

Fig. 5(a) shows the comparison between MC and QMC methods along with the Exact result obtained by using Black Scholes Option Pricing formula. Fig. 5(b) and 5(c) shows the absolute and relative error of MC and QMC methods for a Down-and-Out Barrier Call option pricing. The relative errors from both simulations are reducing with an increase in the number of simulations. The relative errors obtained using the QMC method are generally lesser when compared to that of MC method.

**VIII. Comparison Between MC and QMC Methods for American Put Option Pricing**

We consider a non dividend American put option with  $r=0.04$ ;  $\sigma = 0.2$ ;  $T=1$ ;  $n =10,000$ . By varying  $S_0$  and  $K$ , we get Table 4 by using MC and QMC methods by considering Halton and Sobol sequence.

Since we do not know the analytical solution of American option pricing, we compare the standard error ( $\frac{\sigma}{\sqrt{n}}$ ) of MC and QMC methods which is the statistical convergence rate of the above mentioned method. From the following Table 4, we observe that for higher Strike Prices, both MC and QMC methods gives very similar price for American put Option. On the other hand, for smaller strike price, we observe some fluctuations between MC and QMC methods.

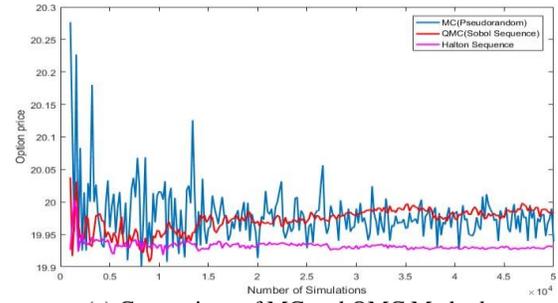
**Table 4. MC and QMC methods for American Put Option Pricing with  $r = 0.04$ ,  $\sigma = 0.2$ ,  $T = 1$  for different  $S_0$  and  $K$ .**

$S_0$	$K$	MC (Standard)	QMC (Halton)	QMC (Sobol)
80	80	5.0521	4.4638	5.1541
	90	11.2444	10.8276	11.2763
	100	19.9902	19.9334	19.9550
	110	29.9208	29.9126	29.9075
85	80	3.3029	2.7160	3.3904
	90	8.0972	7.5011	8.2286
	100	15.5445	15.2388	15.4631
	110	24.9487	24.9114	24.9043
90	80	2.0735	1.6210	2.1534
	90	5.6836	5.0218	5.7984
	100	11.7457	11.2139	11.7958
	110	20.1291	19.9336	20.0162
95	80	1.2536	0.9415	1.3218
	90	3.8982	3.2332	4.0062
	100	8.6792	8.0148	8.8226
	110	15.8798	15.4734	15.8449

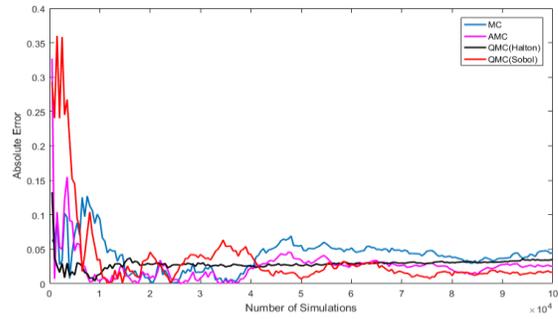
Fig. 6(a) shows the price of a non-dividend american put options with MC (pseudorandom number) and QMC (Halton and Sobol sequence) methods.<sup>6</sup>

As the number of simulations increases, the fluctuation of MC method decreases. For higher number of simulations, both MC and QMC gives the similar results. Fig. 6(b) shows the standard error ( $\frac{\sigma}{\sqrt{n}}$ ) of MC and QMC methods for

American put option pricing where  $\sigma$  is the standard deviation and  $n$  is the number of Simulations. For large number of simulations, the standard error becomes very small for both MC and QMC methods.



(a) Comparison of MC and QMC Method



(b) Standard error of MC and QMC Methods

**Fig. 6.** Comparison and Standard error of MC and QMC methods for American option pricing.

**IX. Conclusion**

In many computational finance problems, asset prices are modeled using the geometric Brownian motion. The simulation of the geometric Brownian motion involves generation of uniform random numbers, sampling from the normal distribution and construction of the Brownian motion paths. We can easily estimate the price of European Options by BS Model. But, for pricing strong path dependent options, there is no other choice but to use Monte Carlo Simulation. We observe that the ordinary MC method underpriced the option in most of the cases, as can be seen in the high discrepancies of the simulated values in comparison to the exact prices. It was also observed that using the AMC method resulted in the reduction of variance estimate when compared to the independent simulations obtained from the standard MC method. This was made possible since the antithetic variables used are negatively correlated and thus, increasing one variable led to a simultaneous decrease in the other variable. Hence, the AMC method gave out a better and fairly accurate result when compared to the MC method. Furthermore, the rate of convergence is accelerated when the simulation nodes are chosen deterministically as observed by the use of low discrepancy sequences. Thus, the use of QMC method together with Halton and Sobol sequences resulted to a more efficient method for valuing complex derivative like Asian, Barrier and American options.

The Monte Carlo convergence rate is  $O(N^{-1/2})$  whereas the quasi-Monte Carlo convergence rate is  $O((\log N)^d/N)$ , where  $N$  is sample size and  $d$  is the dimension of the problem.

Finally, we can conclude that QMC techniques improve the efficiency in most of the cases because standard deviations were always lower for QMC than MC methods. For most of the cases, QMC has also the superior convergence rate compared to MC method.

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