Study of Graded Algebras and General Linear Group with Lie Superalgebras and R-Algebra

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(Received : 22 July 2018 ; Accepted : 7 January 2020)

Abstract

Some elements of theory of \(\mathbb{Z}_2\)-graded rings, modules and algebras, \(\mathbb{Z}_2\)-graded tensor algebra, Lie superalgebras and matrices with entries in a \(\mathbb{Z}_2\)-graded commutative ring are treated in our present paper. At last a Theorem 4.4 on the set of square matrices in the graded R-algebra \(M_R[m|n]\) is established.

Keywords: \(\mathbb{Z}_2\)-graded rings, modules, commutative ring and graded algebras, tensor calculus, general graded linear group \(GL[m|n]\), the set of graded matrices \(M_R[(p+q) \times (m+n)]\) and graded \(R\)-algebra.

I. Introduction

Nowadays a large body of literature is available concerning graded algebras, mainly over the real or complex numbers (usually called superalgebras), their representations, etc. Classical references are [3], [6], [7], [8], [10]. The most common notations and basic results are treated in this article.

II. Graded Algebraic Structures

In general, given an arbitrary group \(G\), we can introduce \(G\)-graded algebraic objects [5], [10]. Since in order to develop a ‘supergeometry’ only \(\mathbb{Z}_2\)-graded structures are needed, we shall only consider here that particular case. We shall assume as a rule that

\[ \text{graded} \equiv \mathbb{Z}_2 - \text{graded} \]

Definition 2.1. A ring \((R, +, \cdot)\) is said to be graded if \((R,+)\) has two subgroups \(R_0\) and \(R_1\) such that \(R = R_0 \oplus R_1\) and \(R_\alpha R_\beta \subseteq R_{\alpha + \beta}\) for all \(\alpha, \beta \in \mathbb{Z}_2\).

An element \(a \in R\) is said to be homogeneous if either \(a \in R_0\) or \(a \in R_1\). On the set \(h(R)\) of homogeneous elements an application \(|\cdot|\) is defined by

\[ |\cdot| : h(R) \rightarrow \mathbb{Z}_2 \]

\[ a \mapsto \alpha \iff a \in R_\alpha. \]

The elements of degree 0 and 1 are called even and odd respectively.

Obviously, any ring \(R\) can be trivially graded: \(R_0 = R\), \(R_1 = \{0\}\).

Example 2.2. Let \(R\) be a \(\mathbb{Z}\)-graded ring, namely, \(R = \bigoplus_{p \in \mathbb{Z}} R_p\) and \(R_p \subseteq R_{p+q}\) then \(R\) can be graded by takig \(R_0\) as the sum of the even components and \(R_1\) as the sum of the odd ones.

For any graded ring \(R\), a graded commutator \(\langle\cdot,\cdot\rangle : R \times R \rightarrow R\) is defined by letting

\[ \langle a, b \rangle = ab - (-1)^{|a||b|}ba \forall a, b \in h(R) \]

The centre of \(R\) is defined as the set

\[ C(R) \equiv \{ a \in R | \langle a, b \rangle = 0 \forall b \in R \}, \]

i.e. \(C(R)\) is the set of the elements of \(R\) which graded – commute with any other elements.

A graded ring \(R\) is said to be graded-commutative if \(\langle a, b \rangle = 0 \forall a, b \in R\), that is, if \(C(R) = R\).

Let \(R\) be a graded ring and \(M\) be a left(right) \(R\)-module.

Definition 2.3. \(M\) is a left (right) graded \(R\)-module if it has two subgroups \(M_0\) and \(M_1\) such that \(M = M_0 \oplus M_1\) and for all \(\alpha, \beta \in \mathbb{Z}_2\), one has \(R_\alpha M_\beta \subseteq M_{\alpha + \beta}\).

If \(R\) is graded-commutative, which we shall henceforth assume, we shall use the term ‘graded \(R\)-module’ without ambiguity.

Having fixed two graded \(R\)-modules \(M\) and \(N\), we say that a morphism \(f : M \rightarrow N\) is \(R\)-linear on the right if \(f(xa) = f(x)a\) for all \(x \in M\) and \(a \in R\). Unless otherwise stated, by ‘linear’ we mean ‘linear on the right’. Moreover, we say that \(f\) has degree \(|f| = \beta \in \mathbb{Z}_2\), if \(f(M_\alpha) \subseteq M_{\alpha + \beta}\) for all \(\alpha \in \mathbb{Z}_2\). The set \(\text{Hom}(M,N)\) of \(R\)-linear morphisms \(M \rightarrow N\) (that will be denoted simply by \(\text{Hom}(M,N)\)) has a natural grading, with \(f \in \text{Hom}(M,N)_\alpha\) whenever \(|f| = \alpha\). If \(R\) is graded-commutative, \(\text{Hom}(M,N)\) is a graded \(R\)-module, with the multiplication rule \((af)(x) = af(x)\).

One of the most basic results in commutative ring theory, namely the Nakayama lemma, can be generalized to the graded setting. Let us define the radical of a graded-commutative ring \(R\) as the graded ideal \(\mathcal{R}\) obtained by intersecting all maximal graded ideals of \(R\).

Proposition 2.4. (Graded Nakayama Lemma) Let \(R\) be a graded-commutative ring \(R\), \(I\) be a graded ideal contained in the radical \(\mathcal{R}\) of \(R\) and \(M\) be a graded finitely generated \(R\)-module.

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(a) If $IM = M$, then $M = 0$.
(b) If $N$ is a graded submodule of $M$ and $M = IM + N$, then $M = N$.
(c) If $x^1, ..., x^m$ are even elements and $y^1, ..., y^n$ are odd elements in $M$ such that the images $(x^1, ..., x^m, y^1, ..., y^n)$ are generators of $M/IM$ over $R/I$, then $(x^1, ..., x^m, y^1, ..., y^n)$ are generators of $M$ over $R$.

Definition 2.5. A graded $R$-module $F$ is said to be free if it has a basis formed by homogeneous elements.

A basis of $F$ of finite cardinality is of type $(m, n)$, if it is formed by $m$ even elements $\{f_{i} \in F| i = 1, ..., m\}$ and $n$ odd elements $\{f_{i} \in F| \alpha = 1, ..., n\}$.

We have a canonical isomorphism

$$F \cong \left( \bigoplus_{i=1}^{m} Rf_{i}^{-1} \right) \oplus \left( \bigoplus_{\alpha = 1}^{n} Rf_{\alpha} \right).$$

For each pair of natural numbers $m, n$ such that $m + n = p$, the $R$-module $R^p$ can be regarded as a free graded $R$-module endowed with a basis of type $(m, n)$, by letting,

$$(R^{m+n})_{0} \equiv R^{m+n} = R_{0} \oplus R_{1}^{m} \oplus (R_{0} \otimes R_{1})^{n};
(R^{m+n})_{\lambda} \equiv R^{m+n} \otimes R_{\lambda}^{m} \oplus R_{0}^{m} \otimes R_{\lambda}^{n} \quad (2.2)$$

$R^{m+n}$ equipped with this gradation will be denoted by $R^{m,n}$.

Example 2.6. (cf. [5]) Let $R$ be a commutative ring, and $M$ be an $R$-module. The exterior algebra of $M$ over $R$, denoted by $\wedge_{R} M$, is a $\mathbb{Z}$-graded algebra, namely $\bigoplus_{p \in \mathbb{Z}} \wedge_{R}^{p} M$, and is alternating, i.e. $x^{2} = 0$ for all $x \in \wedge_{R}^{2p+1} M$. If $M$ is free and finitely generated, with a basis $\{e_{i}| i = 1, ..., N\}$, then $\wedge_{R} M$ is a free finitely generated $R$-module, with a basis (relative to the basis $\{e_{i}\}$) which can be described as follows. Let $\Xi_{N}$ denote the set

$$\left\{ \mu: \{1, ..., r\} \rightarrow \{1, ..., N\}; \text{strictly increasing}; 1 \leq r \leq N \right\} \cup \left\{ \emptyset \right\},$$

where $\emptyset$ is the empty sequence, and let

$$\beta_{\mu} = e_{\mu(1)} \wedge ... \wedge e_{\mu(r)}$$

for $\mu \neq \emptyset$, $\beta_{\emptyset} = 1$.

Then $\{\beta_{\mu}| \mu \in \Xi_{N}\}$ is the canonical basis of $\wedge_{R} M$.

The cases $R = \mathbb{R}$ and $R = \mathbb{C}$ have a particular interest and deserve ad hoc notations:

$$\wedge_{\mathbb{R}} \mathbb{R}^{L} \equiv B_{L}; \wedge_{\mathbb{C}} \mathbb{C}^{L} \equiv C_{L} \quad (2.3)$$

$B_{L}$ is a vector space, with a canonical basis obtained from the canonical basis of $\mathbb{R}^{L}$ according to the above described procedure. If $m_{L}$ is the ideal of nilpotents of $B_{L}$, the vector space direct sum decomposition $B_{L} = \mathbb{R} \oplus m_{L}$ defines two projections

$$\sigma: B_{L} \rightarrow \mathbb{R}; \quad s: B_{L} \rightarrow m_{L} \quad (2.4)$$

which are sometimes called body and soul maps.

Tensor Products: Let us recall that we are considering a graded-commutative ring $R$. The graded tensor product of two graded $R$-modules $M, N$ is by definition the usual tensor product $M \otimes_{R} N$, obtained by regarding $M$ as a right module, and $N$ as a left module, equipped with the gradation

$$(M \otimes_{R} N)_{\gamma} = \bigoplus_{\alpha + \beta = \gamma} m_{\alpha} \otimes n_{\beta} \quad \text{with } m_{\alpha} \in M_{\alpha}, n_{\beta} \in N_{\beta}.$$

Evidently, $M \otimes_{R} N$ has a natural structure of graded $R$-module:

$$a(x \otimes y) = ax \otimes y = (-1)^{|a||x|} xa \otimes y = (-1)^{|a||x|} x \otimes ay = (-1)^{|a||x| + |y|} (x \otimes y)a. \quad (2.5)$$

The graded tensor product can be characterized as a ‘universal object’. To this end, given graded $R$-modules $M, N$ and $Q$, we introduce the set $\mathcal{L}(M, N; Q)_{\alpha}$ (with $\alpha \in \mathbb{Z}_{2}$) of the graded $R$-bilinear morphisms $f: M \times N \rightarrow Q$, homogeneous of degree $\alpha$: if $f \in \mathcal{L}(M, N; Q)_{\alpha}$, then $f$ is a morphism of degree $\alpha$ such that $f(xa, y) = f(x, ay) = (-1)^{|a||x|} f(x, y)a$ for all $a \in R$. The set

$$\mathcal{L}(M, N; Q) \equiv \mathcal{L}(M, N; Q)_{0} \oplus \mathcal{L}(M, N; Q)_{1}$$

is endowed with a structure of graded $R$-module by enforcing the multiplication rule $(fa)(x, y) = f(ax, y)$. In the same way, if $M_{1}, ..., M_{n}, Q$ are graded $R$-modules, we define the graded $R$-module $\mathcal{L}(M_{1}, ..., M_{n}; Q)$ formed by the graded $R$-multilinear morphisms $M_{1} \times \cdots \times M_{n} \rightarrow Q$.

Proposition 2.7. There are natural isomorphisms in the category $R \rightarrow G$ Module

$$\mathcal{L}(M, N; Q) \cong \text{Hom}_{R}(M \otimes_{R} N, Q) \cong \text{Hom}_{R}(M, \text{Hom}_{R}(N, Q)).$$

Proposition 2.8. Let $M, M', M''$ be graded $R$-modules; the following natural isomorphisms of graded $R$-modules hold:

(a) $M \otimes_{R} M' \cong M' \otimes_{R} M$, achieved by the morphism $x \otimes x' \mapsto (-1)^{|x||x'|} x' \otimes x$;
(b) $(M \otimes_{R} M') \otimes_{R} M'' \cong (M' \otimes_{R} M) \otimes_{R} M''$, achieved by the morphism $(x \otimes x') \otimes x'' \mapsto x \otimes (x' \otimes x'')$;
(c) $R \otimes_{R} M = M \cong M \otimes_{R} R$.

If $f: M \rightarrow P$, $g: N \rightarrow Q$ are morphisms of graded modules over a graded ring $R$, the tensor product $f \otimes g: M \otimes_{R} N \rightarrow P \otimes_{R} Q$ is the morphism defined by the condition

$$(f \otimes g)(m \otimes n) = (-1)^{|m||n|} f(m) \otimes g(n). \quad (2.6)$$

III. Graded Algebras and Graded Tensor Calculus

Let $R$ be a graded-commutative ring.

Definition 3.1. A graded $R$-algebra $P$ is a graded $R$-module endowed with a graded $R$-bilinear multiplication

$$P \otimes P ightarrow P$$

$$x \otimes y \mapsto x \cdot y.$$
A graded $R$-algebra $P$ is said to be graded-commutative if all graded commutators
\[ <x, y> = x \cdot y - (-1)^{|x||y|} y \cdot x, \]
defined on the analogy of equation (2.1), vanish.

**Example 3.2.** The graded module $B_{\mathbb{C}}(C_{\mathbb{C}})$ in Example 2.6, equipped with the exterior product, is a graded-commutative $\mathbb{R}$-algebra ($\mathbb{C}$-algebra).

The graded tensor product $P \otimes_R Q$ of two graded $R$-algebras $P$ and $Q$ is defined as the tensor product of the underlying $R$-modules equipped with the multiplication naturally induced by those of $P$ and $Q$
\[ Q: (x_1 \otimes y_1) \cdot (x_2 \otimes y_2) = (-1)^{|y_1||x_2|}(x_1 \cdot x_2) \otimes (y_1 \cdot y_2). \]

**Definition 3.3.** A graded Lie $R$-algebra (or Lie $R$-superalgebra) $\mathfrak{B}$ is a graded $R$-algebra, whose multiplication, called graded Lie bracket and denoted by $[\cdot, \cdot]$, satisfies the following identities:
\[ [x, y] = -(-1)^{|x||y|}[y, x]; \quad (3.1) \]
\[ (-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||z|}[y, [z, x]] + (-1)^{|z||x|}[z, [x, y]] = 0. \]

**Remark 3.4.** Given a graded Lie algebra $\mathfrak{B}$, its even part $\mathfrak{B}_0$ is a Lie algebra over the ring $R_0$.

An important class of graded Lie algebras can be constructed in terms of the notion of graded derivation.

Let $P$ be a graded-commutative $R$-algebra.

**Definition 3.5.** A homogeneous morphism $D \in \text{End}_R P$ is a graded derivation of $P$ over $R$ if it fulfills the following condition (called the graded Leibnitz rule)
\[ D(x \cdot y) = D(x) \cdot y + (-1)^{|x||Dx|}x \cdot D(y). \]

The graded $R$-submodule of $\text{End}_R P$ generated by the graded derivations of $P$ will be denoted by $\text{Der}_R P$, or simply $\text{Der} P$.

**Proposition 3.6.** $\text{Der} P$, equipped with the graded Lie bracket
\[ [D_1, D_2] \equiv D_1 \circ D_2 - (-1)^{|D_1||D_2|} D_2 \circ D_1, \]
(3.4)
is a graded Lie $R$-algebra.

By identifying $R$ with the submodule $R.1 \subset P$, condition (3.4) implies that, for all $D \in \text{Der} P$, $D(R) = 0$. We notice that $\text{Der} P$ is a (left) graded $P$-module in a natural way, by letting $(xD)(y) = x \cdot D(y)$.

**Definition 3.7.** A graded derivation of $P$ over $R$ with values in $M$ is a homogeneous element $D \in \text{Hom}_R(P, M)$ which fulfills a graded Leibnitz rule formally identical with equation (3.3).

The graded $P$-submodule of $\text{Hom}_R(P, M)$ generated by the graded derivations of $P$ with values in $M$ will be denoted by $\text{Der}_R(P, M)$.

**Proposition 3.8.** Let $M$ and $N$ be $R$-modules. There is a natural morphism of graded $R$-modules
\[ \phi: N \otimes M^* \to \text{Hom}(M, N) \]
described by $\phi(n \otimes \omega)(m) = n \omega(m)$. This induces a morphism
\[ \gamma: M^* \otimes N^* \to (M \otimes N)^* \]
whose expression is
\[ \gamma(\omega \otimes \eta)(m \otimes n) = (-1)^{|n||\omega|}(\omega(m) \eta(n)). \]

Both morphisms are bijective whenever $M$ is free and finitely generated.

**Graded Exterior Algebra:** Let $M$ be a graded $R$-module and let us denote by
\[ T^p M = M \otimes \cdots \otimes M \quad (3.1) \]
\[ \text{Th tensor power of } M, \text{ graded as usual.} \]

We consider as in the non-graded setting the graded tensor algebra of $M$,
\[ T(M) = \bigotimes_{p=0}^{\infty} T^p M, \quad (3.5) \]
which is in a natural way a bigraded $R$-algebra (i.e. it has the usual $\mathbb{Z}_2$-gradation of the tensor algebra, together with the $\mathbb{Z}_2$-gradation it carries as a graded $R$-algebra).

The graded exterior algebra $\Lambda_R M$ of $M$ (denoted simply by $\Lambda M$) is defined as the quotient of $T(M)$ by the graded ideal $\mathfrak{I}(M)$ generated by elements of the form
\[ m_1 \otimes m_2 + (-1)^{|m_1||m_2|} m_2 \otimes m_1, \quad \text{with } m_1, m_2 \text{ homogeneous.} \]

The product induced in $\Lambda M$ by this quotient is denoted by $\wedge$ and is called the (graded) wedge product, as usual. If we let $\mathfrak{I}(M) = \mathfrak{I}(M) \cap T^p M$, since $\mathfrak{I}(M)$ is generated by homogeneous elements, we obtain $\mathfrak{I}(M) = \bigotimes_{p=0}^{\infty} \mathfrak{I}(M)$ and therefore,
\[ \wedge M = \bigotimes_{p=0}^{\infty} \wedge^p M \]
with $\wedge^p M = T^p M / \mathfrak{I}(M)$.

We wish to ascertain the relationship existing between the exterior algebra $\Lambda M^*$ and the modules of alternating graded multilinear forms: this will be realized by a morphism analogous to the morphism
\[ y: M^*_1 \otimes \cdots \otimes M^*_n \to (M_1 \otimes \cdots \otimes M_n)^* \approx \mathfrak{L}(M_1, \ldots, M_n; R). \]

If $F_p \in \text{Hom}(T^p M, R)$ and $F_q \in \text{Hom}(T^p M, R)$ are homogeneous graded multilinear forms, $F_p \otimes F_q$ acts on a family of homogeneous elements according to the formula:
Let $S_p$ be the group of permutation of $p$ objects. For any $\sigma \in S_p$ and any $F_p \in \text{Hom}(T^pM, R)$, we write, for homogeneous elements $m_1, \ldots, m_p \in M$,

$$F^\sigma_p(m_1, \ldots, m_p) = (-1)^{\Delta_1(\sigma, m)} F_p(m_{\sigma(1)}, \ldots, m_{\sigma(p)}),$$

where

$$\Delta_1(\sigma, m) = \sum_{1 \leq i < j \leq p} \sum |m_{\sigma(i)}||m_{\sigma(j)}|.$$  \hspace{1cm} (3.7)

**Definition 3.9.** A graded multilinear form $F_p \in \text{Hom}(T^pM, R)$ is said to be alternating if $F^\sigma_p = (-1)^{\Delta_1(\sigma, m)} F_p$ for every $\sigma \in S_p$, where $|\sigma|$ is the parity of the permutation $\sigma$.

The set $\text{Alt}(M \times \cdots \times M; R) \equiv \text{Alt}(M^p, R)$ of all alternating graded multilinear forms is a submodule of $\text{Hom}(T^pM, R)$; we can introduce a projection morphism, which is no more than the graded anti-symmetrization:

$$A_p : \text{Hom}(T^pM, R) \to \text{Alt}(M^p; R),$$

$$F_p \to A_p(F_p) = \frac{1}{p!} \sum_{\sigma \in S_p} (-1)^{|\sigma|} F^\sigma_p.$$  \hspace{1cm} (4.1)

**Proposition 2.10.** The morphism $A_p$ has the following properties:

(a) $A_p(F_p) = F_p$ for any alternating form $F_p$;

(b) $A_{p+q}(F_q \otimes F_q) = (-1)^{pq + |F_q||F_q|} A_{p+q}(F_q \otimes F_q)$ for homogeneous $F_q, F_q$;

(c) $A_{p+q}(A_p(F_q) \otimes F_q) = A_{p+q}(F_q \otimes F_q)$.

We assume that $M$ is a free and finitely generated module, so that we may identify $T^p(M^*)$ with $\text{Hom}(T^pM, R)$. In this way, the morphism $A_p$ yields the exact sequence of graded $R$-modules

$$0 \to \Lambda^p(M^*) \to T^pM^* \to \text{Alt}(M^p; R) \to 0,$$ \hspace{1cm} (3.8)

and therefore we obtain an isomorphism $\Lambda^p M^* \cong \text{Alt}(M^p; R)$. Thus, for a free and finitely generated module $M$, the homogeneous elements in the graded exterior algebra $\Lambda^q M^*$ can be interpreted as alternating graded multilinear forms on $M$. In particular, we may interpret the wedge product of two elements $w^p \in \Lambda^p M^*$ and $w^q \in \Lambda^q M^*$ as a graded multilinear form, which acts on homogeneous elements $m_1, \ldots, m_{p+q}$ according to [9];

$$(\omega^p \wedge \omega^q)(m_1, \ldots, m_{p+q}) = \frac{1}{(p + q)!} \sum_{\sigma \in S_{p+q}} (-1)^{|\sigma| + \Delta_2(\sigma, m, \omega^q)} \omega^p(m_{\sigma(1)}, \ldots, m_{\sigma(p)}) \omega^q(m_{\sigma(p+1)}, \ldots, m_{\sigma(p+q)}),$$

where in terms of the symbol $\Delta_1(\sigma, m)$ previously defined, we get

$$\Delta_2(\sigma, m, \omega^q) = \Delta_1(\sigma, m) + |\omega^q| \sum_{i=1}^{p+q} \sum_{m_{\sigma(i)}} 

\text{IV. Matrices}

Given a graded-commutative ring $R$, an $R$-module morphism $R^n \to R^{|q|}$ can be regarded, relative to the canonical bases of $R^n$ and $R^{|q|}$, as a $(p + q) \times (m + n)$ matrix with entries in $R$.

Let $M_R[(p + q) \times (m + n)]$ be endowed with the so-called graded structure $\omega^p \wedge \omega^q$, and consider the set of matrices of the form (4.1), equipped with this gradation, will be denoted by $M_{R,n}(R^{|q|})$, by decreeing that:

• $X$ is even if $X_1$ and $X_4$ have even entries, while $X_2$ and $X_3$ have odd entries;

• $X$ is odd if $X_1$ and $X_4$ have odd entries, while $X_2$ and $X_3$ have even entries;

The set of matrices of the form (4.1), equipped with this gradation, will be denoted by $M_{R,n}(R^{|q|})$. The set of square matrices $M_{R,n}(R^{|q|})$ which are obtained by letting $p = m$, $q = n$ is a graded $R$-algebra.

The usual notion of trace and determinant of a matrix can be extended to the matrices in $M_{R,n}(R^{|q|})$, thus obtaining the concepts of graded trace and Berezinian (also called supertrace and superdeterminant respectively). For any matrix $X \in M_{R,n}(R^{|q|})$, regarded as a morphism $X^*: (R^{|q|})^* \to (R^{|q|})^*$ dual to $X$. With reference to equation (4.1), one obtains the following relations, where the superscript $t$ denotes the usual matrix transportation:

$$\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}^t = \begin{pmatrix} X_1^t & X_2^t \\ -X_3^t & X_4^t \end{pmatrix} \text{ if } |X| = 0$$

$$\begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}^t = \begin{pmatrix} X_1^t & -X_2^t \\ X_3^t & X_4^t \end{pmatrix} \text{ if } |X| = 1$$

The graded transportation behaves naturally with respect to matrix multiplication:

$$(XY)^t = (-1)^{|X||Y|} Y^t X^t.$$
The graded trace of $X$ is the element $\text{Str}X = \sum a_i^* a^j \in R$. Alternatively, one can give a direct characterization by letting, for all homogeneous $X \in M_R[m|n]$,

$$\text{Str} = \text{Tr}X_1 - (-1)^{|X|}\text{Tr}X_4$$

where $\text{Tr}$ designates the usual trace operation. The graded trace determines an $R$-module morphism $\text{Str}: M_R[m|n] \to R$, which is natural with respect to graded transportation and matrix multiplication:

$$\text{Str}(X^{\alpha}) = \text{Str}X$$

$$\text{Str}(XY) = (-1)^{|X||Y|}\text{Str}(YX).$$

Let us notice that, by denoting by $I_{m|n}$ the identity matrix, one has $\text{Str} I_{m|n} = m - n$.

In order to extend the notion of determinant, we must consider the subgroup $GL_R[m|n]_0$ of the matrices in $M_R[m|n]$ corresponding to an even invertible endomorphisms. $GL_R[m|n]_0$ is the natural extension of the notion of general linear group, so that it will be called the general graded linear group.

**Proposition 4.1.** A matrix $X \in M_R[m|n]_0$ is in $GL_R[m|n]_0$ if and only if $X_1 \in GL_R[m|0]$ and $X_4 \in GL_R[0|n]$, i.e. $X$ is invertible if and only if $X_1$ and $X_4$ are invertible as ordinary matrices with entries in $R_0$.

**Definition 4.2.** [1], [3], [4] Let $X \in GL_R[m|n]$, the Berezinian of $X$ is the element in $GL_R[1|0]$ given by

$$\text{Ber}X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$$

$$= \det(X_1 - X_2X_4^{-1}X_3)(\det X^{-1})$$

**Proposition 4.3.** The mapping $\text{Ber}: GL_R[m|0] \to GL_R[0|n]$ is a group morphism, that coincides with the determinant whenever $n = 0$:

$$\text{Ber}(XY) = \text{Ber}X \text{Ber}Y \ \forall X, Y \in GL_R[m|n]$$

**Theorem 4.4.** A matrix in $X \in M_R[m|n]_0$ is invertible if and only if $\sigma(X) \in GL[m + n]$.

**Proof.** The ‘only if’ part is trivial, since $\sigma$ is ring morphism. To show the converse, it suffices to prove that a matrix $Z \in M_{B_L}[p|0]_0$ is invertible as a matrix with entries in $(B_L)_0$ if $\sigma(Z)$ is invertible. In the case $p = 1$ this is a consequence of the fact that in $B_1$ the morphism $\sigma$ is the natural projection $(B_L)_0 \to (B_L)_0/(n_L)_0$. The result is easily extended to $p > 1$ by inclusion.

**V. Conclusion**

We start with given an arbitrary group $G$ and introducing $G$-graded algebraic objects and for a given graded-commutative ring $R$ and $R$-module morphism can be regarded, relative to the canonical bases of relative to the canonical bases of $R^{m|n}$ and $R^{p|q}$, as a $(p + q) \times (m + n)$ matrix with entries in $R$, $X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}$ which acts on column vectors in $R^{m|n}$ from the left. Finally, this article induces a **Theorem 4.4** on a matrix of graded $R$-algebra. This paper will be helpful for other researchers.

**References**


