

Electromagnetics in Terms of Differential Forms

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Abstract

The calculus of differential forms has been applied to electromagnetic field theory in several papers and texts, some of which are cited in the references. Differential forms are underused in applied electromagnetic research. Differential forms represent unique visual appliance with graphical apprehension of electromagnetic fields. We study the calculus of differential forms and other fundamental principle of electromagnetic field theory. We hope to show in this paper that differential forms make Maxwell's laws and some of their basic applications more intuitive and are a natural and powerful research tool in applied electromagnetics.

Keywords: Maxwell's equations, Differential forms, Exterior derivative.

I. Introduction

Maxwell equations and the medium equation in terms of differential forms, gives the impression that there can't exist a simpler way to express these equations, and so differential forms should serve as a natural language for electromagnetism. However, looking at the literature shows that books and articles are almost exclusively written in Gibbsian vectors. Differential forms have been used by the physicists. The use of the calculus of differential forms in electromagnetics has been explored in several important papers and texts, including Misner, Thorne and Wheeler¹, Deschamps², and Burke³. George Deschamps pioneered the application of differential forms to electrical engineering but never completed his work. A worldwide known authority of differential forms, Ismo V. Lindell gave a feasible idea of classical Gibbsian vector calculus with the mathematical interpretation of differential forms.

James Clerk Maxwell, a famous scientist, discovered the full set of mathematical laws that govern electromagnetic fields. After that, other mathematicians, physicists and engineers have suggested a surprisingly large number of mathematical structures for characterizing fields and waves and performing with electromagnetic theory. Vector notation is the most common in textbooks and engineering work, and has the advantage of being widely taught. The successive development of the vector calculus extended a powerful and useful tool for working with electromagnetic theory. Tensor analysis is actually a brief notation for electromagnetics, but the full description of tensor is typically not needed in application problems. The exterior derivative of differential forms is convenient to work with electromagnetic theory that is simpler than vector analysis or tensor calculus.

Actually, the formulation of differential forms is close to the vector calculus. It is actually analytical tools have been developed, including dydics, bivectors, tensors, quaternions and Clifford algebra. The common theme of all of these mathematical notations is to hide the complexity of the set of 20 coupled differential equations in Maxwell's

original paper⁴ using high level hypothesis for field and source quantities.

For various reasons, higher order mathematical notation are actually used by applied practitioners and engineers. Heaviside's vector analysis is used normally to compare between abstraction and concreteness with dydic notation, adequate for nearly all practical problems.

Differential forms originated in the work of Hermann Günter Grassmann and Élie Cartan. During 1842-43, Grassmann wrote the book *Lineale Ausdehnungslehre*⁵, in which he introduced what is now called exterior algebra. Based on Grassmann's exterior algebra, Cartan⁶ developed the exterior calculus. Differential forms provide direct connection between geometric image and visual interpretation into electromagnetism. Electromagnetic theory combines physical, mathematical and geometric ideas.

In this paper, we represent the various degrees of differential forms, graphical representations of the various field quantities. Maxwell's equations in integral and in point form are also demonstrated. The aim of this paper is to express that differential forms are an attractive and stable to vector analysis for electromagnetic field theory.

II. Differential Forms and Its Degree

The vector concept can be extended by differential forms. The use of differential forms doesn't mean to give up vector concept. When a quantity integrated over integrals, including differentials, the integrated value is called differential forms. $y^2 dx$, $-3 dx dz$ are examples of differential forms. The number of integral in which an object is integrated called for by a differential form decides its degree. The form $y^2 dx$ is integrated under a single integral over a path and so it is a 1-form. The form $-3 dx dz$ is integrated by a double integral over a surface, so its degree is two. A 3-form is integrated by a triple integral over a volume. 0-forms are functions which is integrated by evaluation at a point. Now, we define differential forms of various degrees and identify them with field intensity, flux density, current density, charge density and scalar potential.

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Table 1. Examples of forms of various degrees

Degree	Region of Integration	Example	General Form
0-form	Point	$3y$	$f(x, y, z, \dots)$
1-form	Path	$x^2 dx + y^2 dy$	$\alpha_1 dx + \alpha_2 dy + \alpha_3 dz$
2-form	Surface	$x dydz + y dzdx$	$\beta_1 dydz + \beta_2 dzdx + \beta_3 dx dy$
3-form	Volume	$(x^2 + y^2) dx dy dz$	$g dx dy dz$

Table 2. Representation of field and sources with the differential forms

Quantity	Form	Degree	Units	Vector/Scalar
Electric Field Intensity	E	1-form	V	E
Magnetic Field Intensity	H	1-form	A	H
Electric Flux Density	D	2-form	C	D
Magnetic Flux Density	B	2-form	Wb	B
Electric Current Density	J	2-form	A	J
Electric Charge Density	ρ	3-form	C	q

Different physical properties of the electric and magnetic fields can be represented by 1-form and 2-form. The 1-form represents the energy picture that shows the change in potential energy as a test charge moves across the 1-form surfaces and the 2-form shows the flux of the field that extends from positive charges to negative charges.

Table 3. The representation of the differential forms of electromagnetism which is expanded in components

Differential Forms of Electromagnetism in Component Form
$E = E_x dx + E_y dy + E_z dz$
$H = H_x dx + H_y dy + H_z dz$
$D = D_x dy dz + D_y dz dx + D_z dx dy$
$B = B_x dy dz + B_y dz dx + B_z dx dy$
$J = J_x dy dz + J_y dz dx + J_z dx dy$
$\rho = q dx dy dz$

III. Electromagnetic Fields with Differential Forms

The Scottish physicist, Maxwell combined the mathematical theory of electric and magnetic phenomena. He developed the relationship between electric and magnetic fields by four distinguished equations. From Maxwell's equations in integral form, we can readily determine the degrees of the differential forms that will represent the various field quantities⁷. In vector notation,

$$\oint_P \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_A \mathbf{B} \cdot d\mathbf{A}$$

$$\oint_P \mathbf{H} \cdot d\mathbf{l} = \frac{d}{dt} \int_A \mathbf{D} \cdot d\mathbf{A} + \int_A \mathbf{J} \cdot d\mathbf{A}$$

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V q dv$$

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$$

where A is a surface bounded by a path P , V is a volume bounded by a surface S , q is volume charge density and the

other quantities are defined as usual. By integrating over path, the electric and magnetic field intensity give 1-forms. The electric and magnetic flux densities are integrated over surfaces so that they are 2-forms. Since, electric current density is a 2-form, this is integrated under a surface. The volume charge density is integrated over a volume as it is a 3-form. Table 2 summarizes these forms.

IV. The Exterior Derivative

In this section, we use the exterior derivative operator to express Maxwell's laws as differential equation. The exterior derivative⁸ is a single operator which has the gradient, curl and divergence as special cases, depending on the degree of the differential form on which the exterior derivative has the symbol d , and can be written mathematically as

$$d = \frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy + \frac{\partial}{\partial z} dz \quad (1)$$

Here, the operator "d" operator is similar to a 1-form and the coefficients are partial derivative operators instead of functions. When we use this operator to a differential form, the derivatives replace the coefficients of the form. Also, the differentials combine with those of the form according to the properties of the exterior product.

Exterior Derivative of 0-forms

Let $f(x, y, z)$ be a 0-form. The exterior derivative of 0-form is

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \quad (2)$$

which is a 1-form, the exterior derivative of f . The 1-form df is dual to the gradient of f . The surfaces of the 1-form are equipotentials or level sets of the function f , so that the exterior derivative of a 0-form has a simple graphical interpretation.

Exterior Derivative of 1-forms

The exterior derivative of a 1-form is comparable to the vector curl operator. If H is a random 1-form $H_x dx + H_y dy + H_z dz$, then the exterior derivative of H is

$$\begin{aligned} dH = & \left(\frac{\partial}{\partial x} H_x dx + \frac{\partial}{\partial y} H_x dy + \frac{\partial}{\partial z} H_x dz \right) dx \\ & + \left(\frac{\partial}{\partial x} H_y dx + \frac{\partial}{\partial y} H_y dy \right. \\ & \left. + \frac{\partial}{\partial z} H_y dz \right) dy \\ & + \left(\frac{\partial}{\partial x} H_z dx + \frac{\partial}{\partial y} H_z dy \right. \\ & \left. + \frac{\partial}{\partial z} H_z dz \right) dz \end{aligned}$$

using the antisymmetry of the exterior product, this becomes

$$dH = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) dy dz + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) dz dx + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) dx dy \quad (3)$$

which is a 2-form dual to the curl of the vector field $H_x \hat{x} + H_y \hat{y} + H_z \hat{z}$. The integral of a 1-form over a path is the number of surfaces pierced by the path with respect to the orientation of the surfaces and the direction of integration. This 1-form is physically represented as surfaces in space. The differential $2dz$ produces surfaces perpendicular to the z -axis, as shown in the Figure 1(a). In general, they meet each other, depending on the behavior of the coefficients of the form.

Exterior Derivative of 2-forms

The electric flux density D is a 2-form $D_x dy dz + D_y dz dx + D_z dx dy$. The exterior derivative of a 2-form D is

$$\begin{aligned} dD &= d(D_x dy dz + D_y dz dx + D_z dx dy) \\ &= \left(\frac{\partial}{\partial x} D_x dx + \frac{\partial}{\partial y} D_x dy + \frac{\partial}{\partial z} D_x dz \right) dy dz \\ &\quad + \left(\frac{\partial}{\partial x} D_y dx + \frac{\partial}{\partial y} D_y dy + \frac{\partial}{\partial z} D_y dz \right) dz dx \\ &\quad + \left(\frac{\partial}{\partial x} D_z dx + \frac{\partial}{\partial y} D_z dy + \frac{\partial}{\partial z} D_z dz \right) dx dy \\ &= \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) dx dy dz \end{aligned}$$

where six of the terms vanish due to repeated differentials. The coefficient of the resulting 3-form is the divergence of the vector field dual to D . Graphically, 2-forms are tubes which is shown in the Figure 1(b). As the coefficients of a 2-form increase, the tubes become narrower and more dense. The tubes are oriented in the direction of the associated dual vector. The number of tubes passing through the surface is the integral of a 2-form over the surface which take into account the relative orientation of the tubes and surface.

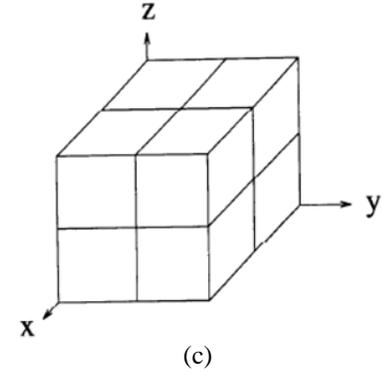
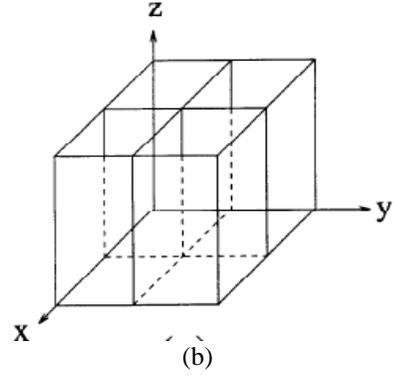
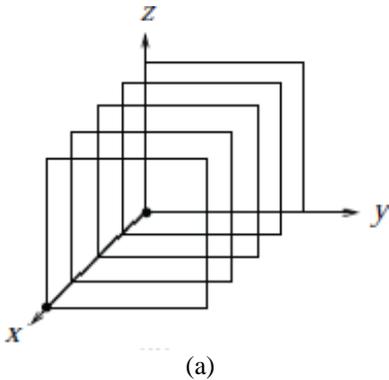


Fig. 1. (a) The 1-form $2dz$, (b) The 2-form $dy dz$, The tubes in the x -direction are formed by the superposition of surfaces of dy and the surfaces of dz . (c) The 3-form $dx dy dz$, with three sets of surfaces that produces boxes.

Exterior Derivative of 3-forms

The exterior derivative of the 3-form ρ is

$$\begin{aligned} d\rho &= d(q dx dy dz) \\ &= \left(\frac{\partial}{\partial x} q dx + \frac{\partial}{\partial y} q dy + \frac{\partial}{\partial z} q dz \right) dx dy dz \\ &= 0 \end{aligned}$$

where the terms vanish due to repeated differentials. Since a 3-form is a volume element, here in the Figure 1(c), this is pictured by boxes. The integral of a 3-form over a volume, where each box is weighted by the sign of the 3-form. A 3-form is dual to its coefficient.

V. The Point Form of Maxwell's Equations

Maxwell's equations in the integral form are given by:

$$\oint_P E = -\frac{d}{dt} \int_A B, \quad \text{Faraday's Law} \quad (4)$$

$$\oint_P H = \frac{d}{dt} \int_A D + \int_A J, \quad \text{Ampere's Law} \quad (5)$$

$$\oint_S D = \int_V \rho, \quad \text{Gauss' Law} \quad (6)$$

$$\oint_S B = 0, \quad \text{Magnetic Flux Continuity} \quad (7)$$

Faraday's law relates the magnetic flux to the electric field. It says that the time derivative of the magnetic flux through

an area A equals the electric tension along the path P . Ampere's law relates current to magnetic field. It states the sum of conduction and displacement current through an area A equals the magnetic tension along the path P . Both equations are tied together via the constitutive equations relating flux densities to field intensities⁸.

Stokes' theorem provides a link between integrals of a differential form and its exterior derivative through the relationship

$$\int_M d\omega = \oint_{\partial M} \omega \quad (8)$$

where M is some region of space and ∂M is its boundary. The dimension of ∂M has to match the degree of ω . If ω is a 0-form, this expression reduces to the fundamental theorem of calculus. If ω is a 1-form, this theorem connects the surface integral of the 2-form $d\omega$, represented by tubes progressing along the surface, to the closed path integral ω , as shown in the Figure 2. This closed path integral is non-zero only if new surfaces of the 1-form are created inside the path. Thus, Stokes' theorem can be interpreted graphically as stating that new surfaces of ω are created by tubes of $d\omega$. If ω is a 2-form, then new tubes of ω extend away from cubes of the 3-form $d\omega$.

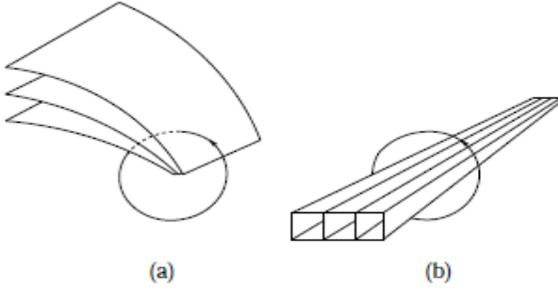


Fig. 2. (a) A nonconsecutive 1-form, with new surfaces extending out into space, allowing a nonzero integral over a closed path. (b) The exterior derivative of the 1-form is a 2-form having tubes where new surfaces of the 1-form are created.

By using Stokes' theorem to the integral form of Maxwell's equations (4) to (7), we obtain Maxwell's equations in the differential forms:

$$dE = -\frac{\partial}{\partial t} B, \quad \text{Faraday's law} \quad (9)$$

$$dH = \frac{\partial}{\partial t} D + J, \quad \text{Ampere's law} \quad (10)$$

$$dD = \rho, \quad \text{Gauss' law} \quad (11)$$

$$dB = 0, \quad \text{Magnetic Flux Continuity} \quad (12)$$

Figure 3 shows the graphical representation of Maxwell's equations. We have two groups of equations, namely Faraday's law and magnetic flux continuity on the left-hand side of the diagram and Ampere's law and Gauss' law⁸ on the right-hand side. The operator exterior derivative is metric-independent operator which enlarges the order of the

corresponding differential forms. The constitutive relations are shown by these groups depending on the coordinate metrics and the elementary properties. These relations are liable for incrementing or decrementing the order of the respective differential forms.

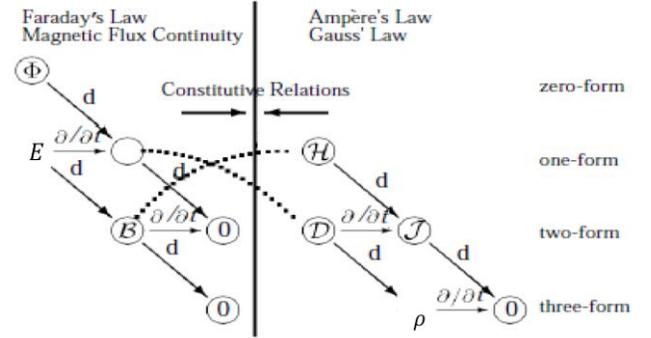


Fig. 3. Graphical representation of Maxwell's equation.

VI. Conclusion

Differential forms make Maxwell's law more intuitive and exterior derivative makes it comfortable to work with. This paper concentrates on the relevance of the exterior calculus to electromagnetics. The simplification found in these areas will likely extend to other problems. Since the calculus of forms can be introduced in the same simple, a combination of differential forms and exterior derivatives could benefit teaching and research in electromagnetics.

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