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An Alternative Approach for Solving Extreme Point Linear and Linear Fractional Programming Problems

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Abstract

The paper considers a class of optimization problems known as extreme point mathematical programming problems. The objective of this paper is to improve the established methods for solving extreme point linear and linear fractional programming problems. To overcome the cumbersome and time consuming procedures of these existing methods, we propose an alternative algorithm to solve such types of problems which is simple and need less computational effort. Two simple examples are given to elucidate our proposed algorithm.

Key words: Linear programming, linear fractional programming, extreme point linear programming (EPLP), extreme point linear fractional programming (EPLFP), simplex method.

I. Introduction

Extreme point mathematical programming is a class of optimization problems in which the objective function (linear or linear fractional) has to be optimized over a convex polyhedron with the additional requirement that the optimal value should exist on an extreme point of another convex polyhedron. A lot of work has been done in extreme point linear programming by Kirby *et al.*[2], Bansal and Bakshi[1]. A number of problems of practical interest can be expressed in the form of extreme point mathematical programming problem. For example, any zero-one integer programming problem can be converted into EPLP by replacing the requirement that each of the variables be either zero or one by the condition that an optimal solution be an extreme point of $\boldsymbol{I_n X} \leq \mathbf{1}, \boldsymbol{X} \geq \mathbf{0}$. Also extreme point technique has been used in solving the fixed charge problem by Puri and Swarup[10]. EPLP first solved by Kirby *et al.*[2], Puri and Swarup[8, 9] developed the techniques which are improvements over the results of Kirby *et al.*[2]. In 1978, Bansal and Bakshi[1] solved this problem using duality relations.

An extreme point linear programming problem can be expressed as

$$\text{Max} \quad Z = \boldsymbol{CX} \tag{1.1}$$

$$\text{Subject to} \quad \boldsymbol{AX} \leq \boldsymbol{b} \tag{1.2}$$

and $\boldsymbol{X}$ is an extreme point of

$$\boldsymbol{DX} \leq \boldsymbol{d} \tag{1.3}$$

$$\boldsymbol{X} \geq \mathbf{0} \tag{1.4}$$

where $\boldsymbol{C}$ is $1 \times n, \boldsymbol{A}$ is $m \times n$, $\boldsymbol{b}$ is $m \times 1, \boldsymbol{D}$ is $p \times n, \boldsymbol{d}$ is $p \times 1$, $\mathbf{0}$ and $\boldsymbol{X}$ are $n \times 1$ real matrices.

For the extreme point linear fractional programming, the objective function will be a ratio of two linear functions like $Q(x) = \frac{P(x)}{D(x)}$. Kirby *et al.*[2]introduced cuts and it generate alternate solutions of $\boldsymbol{DX} \leq \boldsymbol{d}$, $\boldsymbol{X} \geq \mathbf{0}$ which are to be investigated in spite of their known character that they cannot be optimal solutions of the original extreme point linear programming problem. Study of these alternate solutions unnecessarily makes the procedure cumbersome and time consuming. In a paper by Kirby *etal.*[3], various extreme points of $\boldsymbol{DX} \leq \boldsymbol{d}, \boldsymbol{X} \geq \mathbf{0}$ are ranked by enumeration technique where at each stage, we have to consider a new basis for finding the next best extreme point solution. In this approach, procedure starts from a point which is quite far away from the optimal solution of extreme point linear programming problem.

Bansal and Bakshi[1] used duality relations to solve extreme point mathematical programming problem. The developed algorithm studied the sensitivity of the optimal solution of dual of a linear programming problem with respect to the cost of an additional variable with known activity vector and determines this cost in such a way that it gives the optimal value of the given problem.

In this paper, we develop an alternative algorithm for solving both the EPLP and EPLFP. The proposed technique only depends upon the simplex algorithm which is very much different from the techniques developed by Kirby *et al.*[2], Bansal and Bakshi[1] and Puri and Swarup[6]. Here we find all the basic feasible extreme points of the second convex polyhedron $\boldsymbol{DX} \leq \boldsymbol{d}$, $\boldsymbol{X} \geq \mathbf{0}$ using simplex method by considering the problem: Max $Z = \boldsymbol{CX} \left(or\ Q(x) = \frac{P(x)}{Q(x)} \right)$ subject to $\boldsymbol{DX} \leq \boldsymbol{d}$, $\boldsymbol{X} \geq \mathbf{0}$. After checking the feasibility of these extreme points for the original problem, we can find out the optimal solution among these feasible extreme points.

II. Alternative Approach to Solve Extreme Point Linear Programming (EPLP) Problems

Our proposed alternative approach to solve EPLP is based on simplex method. The simplex method is a search procedure that sifts through the basic feasible solutions, one at a time, until the optimal basic feasible solution (whenever it exists) is identified. With m constraints and n variables, the maximum number of basic solutions to the standard linear program is finite and is given by n_{c_m}. By definition, every basic feasible solution is also a basic solution. Hence the maximum number of basic feasible solution is also limited by n_{c_m}. Also if the feasible region is non-empty, closed and bounded, then an optimal to the linear program exists and it is attained at a vertex point of the feasible region (*Extreme point theorem*). On the other hand every vertex of the feasible region corresponds to a basic feasible solution of the problem and vice-versa. This means that an

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optimal solution to a linear program can be obtained by merely examining its basic feasible solutions. This will be a finite process since the number of basic feasible solutions can not exceed n_{c_m}. The simplex method will begin the search at (any) one of the vertices and then ascend, as if we are climbing a hill, toward the optimal vertex along the edges of the feasible region. Since two or more edges of the feasible region meet at a vertex, we will have two or more path to reach the optimal vertex. By considering all these paths, we will have all the basic feasible solutions from the simplex tableau.

The algorithm can be summarized in the following basic steps:

1. Consider the problem: Max $Z = \boldsymbol{CX}$
   Subject to $\boldsymbol{DX} \leq \boldsymbol{d}, \boldsymbol{X} \geq \boldsymbol{0}$.
2. Find all basic feasible solutions using simplex method by taking all possible entering variables under consideration.
3. Check the feasibility of these obtained extreme points for the original constraint set.
4. Find out the optimal solution among these feasible extreme points.

III. Notations

Now first we consider the following problem instead of the problem (1.1)-(1.4).

$$\left.\begin{array}{ll} Max & Z = \boldsymbol{CX} \\ Subject\ to & \boldsymbol{DX} \leq \boldsymbol{d} \\ & \boldsymbol{X} \geq \boldsymbol{0} \end{array}\right\} \quad (T)$$

Let

$D_1 =$ Set of all decision variables.

$X_i =$ Set of all extreme points of the feasible region corresponding to all basic feasible solutions of (T) with initial entering variable x_i into the basis till the end of all iterations including initial basic feasible solution.

$E = \cup_{i=1}^{n} X_i =$ Set of all extreme points of the feasible region of (T).

$S_0 = \{\boldsymbol{X} \in E | \boldsymbol{X}$ is not feasible for the original problem

$$\left.\begin{array}{ll} \text{Max} & Z = \boldsymbol{CX} \\ \text{Subject to} & \boldsymbol{AX} \leq \boldsymbol{b} \\ & \boldsymbol{X} \geq \boldsymbol{0} \end{array}\right\} \quad (M)$$

$S_1 = E \backslash S_0$

$Z_{max} = Max\{\boldsymbol{CX}: \boldsymbol{X} \in S_1\}$

IV. Algorithm

Our proposed algorithm can be summarized in the following steps:

Step 1:Solve the problem (T) by using simplex method with entering variable $x_i, i \in \{1,2, \dots, n\}$ and then obtain X_i.

Set $D_2 = \{x_i | x_i\ correspond\ to\ X_i\}$. Set $D_0 = D_1 \backslash D_2$.

Step 2: If $D_0 \neq \emptyset$, go to step 1. Otherwise go to Step 3.

Step 3: Set $E = \cup_{i=1}^{n} X_i$

Step 4: Check whether each $\boldsymbol{X} \in E$ is feasible or not for the problem (M) to obtain S_0.

Step 5:Set$S_1 = E \backslash S_0$.

Step 6: Calculate the value of Z at each extreme point $\boldsymbol{X} \in S_1$ and determine the optimal value of the objective function among these values of Z.

Step 7: Say, Z is optimal at $\boldsymbol{X_0}$ and the optimal value is Z_{max}.

The use of the algorithm is now demonstrated with the following two examples in which the first one is from Kirby *et al.*[2] and the last one is from Puri and Swarup[6].

Example I:

Max $Z = x_1 + 20x_2$

Subject to $x_1 + x_2 \leq 11$

$3x_1 + 5x_2 \leq 45$

and (x_1, x_2) is an extreme point of

$-5x_1 + x_2 \leq 1$

$2x_1 + x_2 \leq 22$

$x_1, x_2 \geq 0$

Consider the following problem

Max $Z = x_1 + 20x_2$

Subject to $-5x_1 + x_2 \leq 1$

$2x_1 + x_2 \leq 22$

$x_1, x_2 \geq 0$

Introduce the slack variables x_3, x_4 to obtain the standard form as,

Max $Z = x_1 + 20x_2$

Subject to $-5x_1 + x_2 + x_3 = 1$

$2x_1 + x_2 + x_4 = 22$

$x_1, x_2, x_3, x_4 \geq 0$

In this problem we have, $D_1 = \{x_1, x_2\}$

Now we can apply the simplex method to solve the problem and we get the following simplex tableau:

Tableau 1

C_B	Basis \ c_j	1	20	0	0	Const.
		x_1	x_2	x_3	x_4	
0	x_3	-5	**1**	1	0	1 ←
0	x_4	2	1	0	1	22
$\bar{c_j} = c_j - z_j$		1	20 ↑	0	0	$Z = 0$

Here(0,0) is an extreme point corresponding to initial basic feasible solution of (T) and consider x_2 as an initial entering variable. So we have$D_2 = \{x_2\}\ and\ X_2 = \{(0,0)\}$.

Next tableau becomes

Tableau 2

C_B	Basis \ c_j	1	20	0	0	Const.
		x_1	x_2	x_3	x_4	
20	x_2	-5	1	1	0	1
0	x_4	**7**	0	-1	1	21 ←
$\overline{c_j} = c_j - z_j$		101	0 ↑	-20	0	$Z = 20$

From Tableau 2, we get (0,1) as an extreme point and thus X_2 becomes as

$$X_2 = \{(0,0),(0,1)\}.$$

We have the next tableau as,

Tableau 3

C_B	Basis \ c_j	1	20	0	0	Const.
		x_1	x_2	x_3	x_4	
20	x_2	0	1	2/7	5/7	16
1	x_1	1	0	-1/7	1/7	3
$\overline{c_j} = c_j - z_j$		0	0	$-\frac{39}{7}$	$-\frac{101}{7}$	$Z = 323$

which is an optimal tableau gives an extreme point (3,16) and thus X_2 becomes

$$X_2 = \{(0,0),(0,1),(3,16)\}.$$

Now from Tableau 1, we see that x_1 can also be taken as initial entering variable as follows,

Tableau 1

C_B	Basis \ c_j	1	20	0	0	Const.
		x_1	x_2	x_3	x_4	
0	x_3	-5	1	1	0	1
0	x_4	**2**	1	0	1	22 ←
$\overline{c_j} = c_j - z_j$		1 ↑	20	0	0	$Z = 0$

So we have $D_2 = \{x_2, x_1\}$ and $X_1 = \{(0,0)\}$.

Next tableau becomes

Tableau 4

C_B	Basis \ c_j	1	20	0	0	Const.
		x_1	x_2	x_3	x_4	
0	x_3	0	**7/2**	1	5/2	56 ←
1	x_1	1	1/2	0	1/2	11
$\overline{c_j} = c_j - z_j$		0	39/2 ↑	0	-1/2	$Z = 11$

From Tableau 4, we get (11,0) as an extreme point and thus X_1 becomes as

$$X_1 = \{(0,0),(11,0)\}.$$

We get the next tableau as,

Tableau 5

C_B	Basis \ c_j	1	20	0	0	Const.
		x_1	x_2	x_3	x_4	
0	x_2	0	1	2/7	5/7	16
1	x_1	1	0	-1/7	1/7	3
$\overline{c_j} = c_j - z_j$		0	0	$-\frac{39}{7}$	$-\frac{101}{7}$	$Z = 323$

Which is an optimal tableau gives an extreme point (3, 16) and thus X_1 becomes

$$X_1 = \{(0,0),(11,0),(3,16)\}.$$

Now we have that $D_0 = D_1 \backslash D_2 = \emptyset$, so we stop the iteration and we get

$$\begin{aligned} E &= \bigcup_{i=1}^{2} X_i \\ &= X_1 \cup X_2 \\ &= \{(0,0),(11,0),(3,16)\} \cup \{(0,0),(0,1),(3,16)\} \\ &= \{(0,0),(11,0),(0,1),(3,16)\} \end{aligned}$$

Now we can check the feasibility of the obtained extreme points as follows:

Extreme points (x_1, x_2)	Constraints $AX \le b$	Status (feasible/ infeasible)	Value of Z
(0,0)	$x_1 + x_2 \le 11 \Rightarrow 0 + 0 = 0 \le 11$ $3x_1 + 5x_2 \le 45 \Rightarrow 3(0) + 5(0) = 0 \le 45$	Feasible	$Z = 0$
(11,0)	$x_1 + x_2 \le 11 \Rightarrow 11 + 0 = 11 = 11$ $3x_1 + 5x_2 \le 45 \Rightarrow 3(11) + 5(0) = 33 \le 45$	Feasible	$Z = 11$
(0,1)	$x_1 + x_2 \le 11 \Rightarrow 0 + 1 = 1 \le 11$ $3x_1 + 5x_2 \le 45 \Rightarrow 3(0) + 5(1) = 5 \le 45$	Feasible	$Z = 20$
(3,16)	$x_1 + x_2 \le 11 \Rightarrow 3 + 16 = 19 \nleq 11$ $3x_1 + 5x_2 \le 45 \Rightarrow 3(3) + 5(16) = 89 \nleq 45$	Infeasible	

From the above table, we get, $S_0 = \{(3,16)\}$.

$\therefore S_1 = E \backslash S_0 = \{(0,0),(11,0),(0,1)\}$ and Z_{max}

$= \max\{0,11,20\} = 20.$

So the optimal solution of the Example I is $x_1 = 0, x_2 = 1$ and the optimal value of the objective function is $Z_{max} = 20.$

V. Extreme Point Linear Fractional Programming (EPLFP) Problem

We can use the same algorithm, described in the section IV, to solve an extreme point linear fractional programming problem using simplex method of Martos[5].

To demonstrate the algorithm, consider the EPLFP problem from Puri and Swarup[6] which is given below.

Example II:

$$\left.\begin{aligned} Max \quad & Q(x) = \frac{2x_1 + x_2}{4x_1 + x_2 + 1} \\ Subject\ to \quad & -2x_1 + x_2 \le 1 \\ & 2x_1 + 5x_2 \le 23 \\ & 2x_1 + x_2 \le 15 \end{aligned}\right\} \quad (5.1)$$

and $(x_1, x_2)$is an extreme point of

$$\begin{gathered} -3x_1 + 2x_2 \le 4 \\ x_1 + 4x_2 \le 22 \\ 5x_1 + 4x_2 \le 46 \\ x_1 - 2x_2 \le 5 \\ x_1, x_2 \ge 0 \end{gathered}$$

Consider the following linear fractional programming problem

$$\left.\begin{aligned} Max \quad & Q(x) = \frac{2x_1 + x_2}{4x_1 + x_2 + 1} \\ subject\ to \quad & -3x_1 + 2x_2 \le 4 \\ & x_1 + 4x_2 \le 22 \\ & 5x_1 + 4x_2 \le 46 \\ & x_1 - 2x_2 \le 5 \\ & x_1, x_2 \ge 0 \end{aligned}\right\} \quad (5.2)$$

Introduce the slack variables x_3, x_4, x_5, x_6 to obtain the standard form as,

$$\left.\begin{aligned} Max \quad & Q(x) = \frac{2x_1 + x_2}{4x_1 + x_2 + 1} \\ Subject\ to \quad & -3x_1 + 2x_2 + x_3 = 4 \\ & x_1 + 4x_2 + x_4 = 22 \\ & 5x_1 + 4x_2 + x_5 = 46 \\ & x_1 - 2x_2 + x_6 = 5 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \ge 0 \end{aligned}\right\} \quad (5.3)$$

In this problem we have, $D_1 = \{x_1, x_2\}$.

Now we can apply the simplex method to solve the problem (5.2) and we get the following simplex tableau:

Tableau 1

P_B	D_B	Basis \ p_j / d_j	x_1	x_2	x_3	x_4	x_5	x_6	Const.
		p_j	2	1	0	0	0	0	
		d_j	4	1	0	0	0	0	
0	0	x_3	-3	2	1	0	0	0	4
0	0	x_4	1	4	0	1	0	0	22
0	0	x_5	5	4	0	0	1	0	46
0	0	x_6	**1**	-2	0	0	0	1	5 ←
$P(x) = 0$		Δ_j'	2	1	0	0	0	0	$Q(x) = 0$
$D(x) = 1$		Δ_j''	4	1	0	0	0	0	
$\Delta_j = \Delta_j' - Q(x)\Delta_j''$			2 ↑	1	0	0	0	0	

Here x_1 is an initial entering variable and (0,0) is an extreme point of the feasible region defined by the constraints of (5.3). So we have$D_2 = \{x_1\}$ and $X_1 = \{(0,0)\}$. Next tableau becomes

Tableau 2

P_B	D_B	Basis \ p_j / d_j	x_1	x_2	x_3	x_4	x_5	x_6	Const.
		p_j	2	1	0	0	0	0	
		d_j	4	1	0	0	0	0	
0	0	x_3	0	-4	1	0	0	3	19
0	0	x_4	0	6	0	1	0	-1	17
0	0	x_5	0	**14**	0	0	1	-5	21 ←
2	4	x_1	1	-2	0	0	0	1	5
$P(x) = 10$		Δ_j'	0	5	0	0	0	-2	$Q(x) = \frac{10}{21}$
$D(x) = 21$		Δ_j''	0	9	0	0	0	-4	
$\Delta_j = \Delta_j' - Q(x)\Delta_j''$			0	5/7 ↑	0	0	0	-2/21	

From Tableau 2, we get (5,0) as an extreme point of the feasible region defined by constraints of (5.3) and thus X_1 becomes $X_1 = \{(0,0), (5,0)\}$.The next tableau becomes,

Tableau 3

		p_j	2	1	0	0	0	0	
		d_j	4	1	0	0	0	0	
P_B	D_B	Basis	x_1	x_2	x_3	x_4	x_5	x_6	Const.
0	0	x_3	0	0	1	0	2/7	11/7	25
0	0	x_4	0	0	0	1	-3/7	**8/7**	8 ←
1	1	x_2	0	1	0	0	1/14	-5/14	3/2
2	4	x_1	1	0	0	0	1/7	2/7	8
$P(x)=\frac{35}{2}$		Δ'_j	0	0	0	0	-5/14	-3/14	$Q(x)=\frac{35}{69}$
$D(x)=\frac{69}{2}$		Δ''_j	0	0	0	0	-9/14	-11/14	
$\Delta_j=\Delta'_j-Q(x)\Delta''_j$			0	0	0	0	-5/161	89/483	
								↑	

From Tableau 3, we get $(8,\frac{3}{2})$ as an extreme point and thus X_1 becomes as, $X_1=\left\{(0,0),(5,0),(8,\frac{3}{2})\right\}$. The next tableau becomes

Tableau 4

		p_j	2	1	0	0	0	0	
		d_j	4	1	0	0	0	0	
P_B	D_B	Basis	x_1	x_2	x_3	x_4	x_5	x_6	Const.
0	0	x_3	0	0	1	-11/8	**7/8**	0	14 ←
0	0	x_6	0	0	0	7/8	-3/8	1	7
1	1	x_2	0	1	0	5/16	-1/16	0	4
2	4	x_1	1	0	0	-1/4	1/4	0	6
$P(x)=16$ $D(x)=29$		Δ'_j	0	0	0	3/16	-7/16	0	$Q(x)=\frac{16}{29}$
		Δ''_j	0	0	0	11/16	-15/16	0	
$\Delta_j=\Delta'_j-Q(x)\Delta''_j$			0	0	0	-89/464	37/464	0	
							↑		

From Tableau 4, we get (6,4) as an extreme point and thus X_1 becomes as $X_1=\{(0,0),(5,0),\left(8,\frac{3}{2}\right),(6,4)\}$.we get the next tableau as,

Tableau 5

		p_j	2	1	0	0	0	0	
		d_j	4	1	0	0	0	0	
P_B	D_B	Basis	x_1	x_2	x_3	x_4	x_5	x_6	Const.
0	0	x_5	0	0	8/7	-11/7	1	0	16
0	0	x_6	0	0	3/7	2/7	0	1	13
1	1	x_2	0	1	1/14	3/14	0	0	5
2	4	x_1	1	0	-2/7	**1/7**	0	0	2 ←
$P(x)=9$ $D(x)=14$		Δ'_j	0	0	1/2	-1/2	0	0	$Q(x)=\frac{9}{14}$
		Δ''_j	0	0	15/14	-11/14	0	0	
$\Delta_j=\Delta'_j-Q(x)\Delta''_j$			0	0	-37/196	1/196	0	0	
						↑			

From Tableau 5, we get (2,5) as an extreme point and thus X_1 becomes as $X_1=\{(0,0),(5,0),\left(8,\frac{3}{2}\right),(6,4),(2,5)\}$.we get the next tableau as,

Tableau 6

P_B	D_B	p_j / d_j / Basis							Const.
		p_j	2	1	0	0	0	0	
		d_j	4	1	0	0	0	0	
		Basis	x_1	x_2	x_3	x_4	x_5	x_6	
0	0	x_5	11	0	-2	0	1	0	38
0	0	x_6	-2	0	1	0	0	1	9
1	1	x_2	-3/2	1	½	0	0	0	2
0	0	x_4	7	0	-2	1	0	0	14
$P(x) = 2$, $D(x) = 3$		Δ'_j	7/2	0	-1/2	0	0	0	$Q(x) = \frac{2}{3}$
		Δ''_j	11/2	0	-1/2	0	0	0	
$\Delta_j = \Delta'_j - Q(x)\Delta''_j$			-1/6	0	-1/6	0	0	0	

which is an optimal tableau gives an extreme point $(0,2)$ and thus X_1 becomes as

$$X_1 = \{(0,0), (5,0), \left(8, \frac{3}{2}\right), (6,4), (2,5), (0,2)\}.$$

Now from Tableau 1, we see that x_2 can also be taken as initial entering variable as follows,

Tableau 1

P_B	D_B	p_j / d_j / Basis							Const.
		p_j	2	1	0	0	0	0	
		d_j	4	1	0	0	0	0	
		Basis	x_1	x_2	x_3	x_4	x_5	x_6	
0	0	x_3	-3	**2**	1	0	0	0	4 ←
0	0	x_4	1	4	0	1	0	0	22
0	0	x_5	5	4	0	0	1	0	46
0	0	x_6	1	-2	0	0	0	1	5
$P(x) = 0$, $D(x) = 1$		Δ'_j	2	1	0	0	0	0	$Q(x) = 0$
		Δ''_j	4	1	0	0	0	0	
$\Delta_j = \Delta'_j - Q(x)\Delta''_j$			2	1 ↑	0	0	0	0	

So we have $D_2 = \{x_1, x_2\}$ and $X_2 = \{(0,0)\}$. Next tableau becomes

Tableau 7

P_B	D_B	p_j / d_j / Basis							Const.
		p_j	2	1	0	0	0	0	
		d_j	4	1	0	0	0	0	
		Basis	x_1	x_2	x_3	x_4	x_5	x_6	
1	1	x_2	-3/2	1	1/2	0	0	0	2
0	0	x_4	7	0	-2	1	0	0	14
0	0	x_5	11	0	-2	0	1	0	38
0	0	x_6	-2	0	1	0	0	1	9
$P(x) = 2$, $D(x) = 3$		Δ'_j	7/2	0	-1/2	0	0	0	$Q(x) = \frac{2}{3}$
		Δ''_j	11/2	0	-1/2	0	0	0	
$\Delta_j = \Delta'_j - Q(x)\Delta''_j$			-1/6	0	-1/6	0	0	0	

which is an optimal tableau gives an extreme point $(0,2)$ and thus X_2 becomes as

$$X_2 = \{(0,0), (0,2)\}$$

Now we have that $D_0 = D_1 \backslash D_2 = \emptyset$, so we stop the iteration and we get

$$E = \bigcup_{i=1}^{2} X_i = X_1 \cup X_2$$
$$= \{(0,0),(5,0),\left(8,\frac{3}{2}\right),(6,4),(2,5),(0,2)\} \cup \{(0,0),(0,2)\}$$
$$= \{(0,0),(5,0),\left(8,\frac{3}{2}\right),(6,4),(2,5),(0,2)\}$$

Now we can check the feasibility of the obtained extreme points as follows:

Extreme points (x_1, x_2)	Constraints $AX \le b$	Status (feasible/ infeasible)	Value of $Q(x)$
(0,0)	$-2x_1 + x_2 \le 1 \Rightarrow -2(0)+0 = 0 \le 1$ $2x_1 + 5x_2 \le 23 \Rightarrow 2(0)+5(0) = 0 \le 23$ $2x_1 + x_2 \le 15 \Rightarrow 2(0)+0 = 0 \le 15$	Feasible	$Q(x) = 0$
(5,0)	$-2x_1 + x_2 \le 1 \Rightarrow -2(5)+0 = -10 \le 1$ $2x_1 + 5x_2 \le 23 \Rightarrow 2(5)+5(0) = 10 \le 23$ $2x_1 + x_2 \le 15 \Rightarrow 2(5)+0 = 10 \le 15$	Feasible	$Q(x) = \frac{10}{21}$
$(8,\frac{3}{2})$	$-2x_1 + x_2 \le 1 \Rightarrow -2(8)+\frac{3}{2} = -\frac{29}{2} \le 1$ $2x_1 + 5x_2 \le 23 \Rightarrow 2(8)+5\left(\frac{3}{2}\right) = \frac{47}{2} \nleq 23$ $2x_1 + x_2 \le 15 \Rightarrow 2(8)+\frac{3}{2} = \frac{35}{2} \nleq 15$	Infeasible	
(6,4)	$-2x_1 + x_2 \le 1 \Rightarrow -2(6)+4 = -8 \le 1$ $2x_1 + 5x_2 \le 23 \Rightarrow 2(6)+5(4) = 32 \nleq 23$ $2x_1 + x_2 \le 15 \Rightarrow 2(6)+4 = 16 \nleq 15$	Infeasible	
(2,5)	$-2x_1 + x_2 \le 1 \Rightarrow -2(2)+5 = 1 = 1$ $2x_1 + 5x_2 \le 23 \Rightarrow 2(2)+5(5) = 29 \nleq 23$ $2x_1 + x_2 \le 15 \Rightarrow 2(2)+5 = 9 \le 15$	Infeasible	
(0,2)	$-2x_1 + x_2 \le 1 \Rightarrow -2(0)+2 = 2 \nleq 1$ $2x_1 + 5x_2 \le 23 \Rightarrow 2(0)+5(2) = 10 \le 23$ $2x_1 + x_2 \le 15 \Rightarrow 2(0)+2 = 2 \le 15$	Infeasible	

From the above table, we get, $S_0 = \left\{\left(8,\frac{3}{2}\right),(6,4),(2,5),(0,2)\right\}$

$\therefore S_1 = E \setminus S_0 = \{(0,0),(5,0)\}$ and $Q_{max} = \max\{0,\frac{10}{21}\} = \frac{10}{21}$.

So the optimal solution of the *Example II* is $x_1 = 5, x_2 = 0$ and the optimal value of the objective function is $Q_{max} = 20$ which is exactly same as obtained by solving using Puri and Swarup[6] method.

VI. Computational Comparison

- The simplex tableau in the procedure by Kirby *et al.*[2]contains more variables as well as constraints (the given *example I* contains 6 variables and 4 constraints) than the simplex tableau in the procedure proposed by us (contains 4 variables and 2 constraints).
- The simplex tableau of the given *example II* of the method of Puri and Swarup[6] contains 9 variables and 7 constraints which is difficult and time consuming to solve by hand calculation where as the tableau in our method contains 6 variables and 4 constraints. Moreover their method needs more algebraic calculation at each iteration.
- The methods of Kirby *et al.*[2] ,Bansal and Bakshi[1] , Puri and Swarup[6]consider the constraints $\boldsymbol{AX} \le \boldsymbol{b}$ and $\boldsymbol{DX} \le \boldsymbol{d}$ simultaneously. As a result the simplex tableau of their methods becomes more complicated. Whereas we first consider only the additional constraints $\boldsymbol{DX} \le \boldsymbol{d}$, which make our simplex tableau more simple, for extreme points and then we check the feasibility of these points for the original constraints $\boldsymbol{AX} \le \boldsymbol{b}$.

All of these provide that our proposed algorithm needs less computational effort to solve EPLP and EPLFP problems because the efficiency of the simplex method depends on the number of iterations (which depend on number of constraints and variables) before reaching the optimal solution.

VII. Conclusion

In this paper, an alternative algorithm has been developed to solve both the EPLP and EPLFP problems based on simplex method of Dantzig[4] and Martos[5]which is simple and needs less computational effort than the methods of Kirby *et al.*[2], Banshal and Bakshi [1], Puri and Swarup[6] to obtain the optimal solution.

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