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Numerical Approximation of a Linear Parabolic PDE

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Abstract

We study stability and convergence analysis of a finite difference approximation of a linear parabolic partial differential equation (PDE) in a periodic domain. In particular, we analyze consistency, stability and order of accuracy of a central space forward time scheme to solve the PDE.

Keywords: Stability; accuracy; convergence.

I. Introduction

Various problems from chemical, physical, mechanical, biological and many other applied sciences have been modeled by reaction diffusion systems as well as by advection reaction diffusions systems. There are various such models that contain local [8, 12] or nonlocal diffusion [2, 4, 5, 7] operators and many contain both [6]. These types of models are typically complicated, interesting to scientists, challenging to understand substantially and to analyze. In general, these type of models can be written as

$$u_t(x,t) = au_x(x,t) + \varepsilon\big(J(x) * u(x,t) - u(x,t)\ \big) + \big(x,t)\big) + f(u),\ x \in \Re, \tag{1}$$

where $\alpha \in R$, $\beta < 0$, $\varepsilon \geq 0$ are constants, J(x) is a kernel function representing special behavior of the problem modeled, f(u) is a nonlinear function, and $J(x) * u\ (x,t)$ is a convolution represented by

$$J(x) * u\ (x,t) == \int_{\Omega} J\ (x-y)\ u\ (y,t)dy.$$

In [11], the author discussed various issues of finite difference approximations of partial differential equations (PDE) in an infinite domain. He discussed welposedness, stability, accuracy and convergence of various finite difference approximations of time dependent PDE. In [8], the author analyzed accuracy of Crank-Nicolson and Richtmyer-Morton methods for local diffusion and advection operators considering a non-periodic domain. In [9], the author discussed finite difference schemes for linear variable coefficient diffusion operators.

In [7], the authors study the model (1) considering $\alpha = 0$, $\beta = 0$: They show coarsening of solutions, numerical approximations of the problem. Accuracy of any such approximation is also discussed in [7] in short.

In [10] and [12] authors study spectral methods for parabolic problems. In particular, the authors in [10], restrict themselves with the stability issues of the Fourier spectral method. A simple one step approximation of a linear partial integro-differential equation is well studied in [3]. The author analyzed stability and accuracy conditions as well as the rate of convergence considering both smooth and non-smooth initial functions.

To be precise, we consider the linear part of (1) with $\in = 0$ (f (u) = 0) which contains advected local diffusion operator (which is related to the famous Fokker- Planck equation [11]) here. Instead of considering the problem as a Cauchy or a Neumann problem in a bounded domain, we consider it as an initial boundary value problem (IBVP) in a periodic domain and analyze the stability and convergence of the scheme used to solve the IBVP. Finite difference methods are used here to approximate solutions of partial differential equations as it is required in several practical situations. There are various highly accurate schemes like pseudo-spectral method, Fourier transform techniques, higher order piecewise polynomial schemes to approximate (1). Here we restrict our self with the stability and convergence analysis of a finite difference scheme in both space and time. We use discrete and continuous Fourier transforms in our stability and convergence analysis. As we consider the problem in a periodic domain we link the discrete Fourier transform results in a periodic domain and the continuous Fourier transform using Poisson summation formula. We study the famous book [1] for the definitions and theorems about Sobolev space in a periodic domain which has been extensively used in our convergence analysis.

We organize this paper in the following way. In Section 2, we present the Problem and it's discretisation in both space and time. In Section 3, we show consistency and stability of the scheme, whereas Section 4 deals with accuracy of the scheme used to generate approximate solutions.

II. The Problem and Finite Difference Approximation

We consider the following IBVP

$$Lu(x,t) = u_t(x,t) - \alpha u_x(x,t) - \beta u_{xx}(x,t) = 0 \tag{2}$$

in a periodic spatial domain [0; 2L] i.e., the boundary condition is $u(x, t) = u(x+2kL, t)$ for all $k \in Z$ with a given initial function u(x, 0) = u_0 : We approximate (2) in space by

$$\frac{du_i}{dt} = \alpha\frac{u_{i+1} - u_{i-1}}{2\Delta x} + \beta\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} \qquad (3)$$

For all i = 1, 2, 3, ..., N where Δx is an uniform space mesh and $u_i = u(x_{i,} t)$. (3) is now semidiscrete time dependent ordinary differential equations (ODE) and can be solved by any one or multistep explicit or implicit schemes. Here we use the explicit Euler scheme to integrate the time dependent system of ODEs. Using the explicit Euler scheme for the time integration of (3) we get

$$U_i^{n+1} = U_i^n + \alpha\frac{\Delta t}{2\Delta x}\left(U_{i+1}^n - U_{i-1}^n\right) + \beta\frac{\Delta t}{\Delta x^2}\left(U_{i+1}^n - 2U_i^n + U_{i-1}^u\right) \qquad (4)$$

where Δt is the time step length and $U_i^n = u(x_i, t_n)$.

III. Stability of the Approximation

In this section we discuss stability of the scheme (4). It is very straight-forward to verify that the scheme used to solve (2) is consistent. Thus we motivate ourselves to discuss stability of the scheme (4). Throughout the stability, accuracy and convergence analysis we apply the Fourier series and the Fourier transform definitions and theorems. Before getting into the main discussion let us introduce some necessary definitions and theorems which are discussed in detail in [13].

Definition 1. The Discrete Fourier Transform for a periodic function (DFTPF) can be defined by $\widetilde{U}_k = h\sum_{j=0}^{N-1} u(x_j)e^{\frac{-i\pi k x_j}{L}}$ for all k = 0, 1, 2, ..., N -1 or k = $-\frac{N}{2}+1, \frac{-N}{2}, ..., \frac{N}{2}$ 2 where u(x, t) is a periodic function with period L and its inverse Fourier transform is defined by $\sum_{k=0}^{N-1}\widetilde{U}_j e^{\frac{i\pi k x_j}{L}}$ or $u_i = \sum_{k=-\frac{n}{2}+1}^{\frac{n}{2}}\widetilde{U}_j e^{\frac{i\pi k x_j}{L}}$.

Definition 2. For any periodic function f(x) with period 2L, its Fourier series can be defined by

$$f(x) = \sum_{n=-\infty}^{\infty} F_n e^{\frac{i\pi n x}{L}}$$

where $F_n = \frac{1}{2L}\int_0^{2L} f(x)e^{\frac{-i\pi k x_j}{L}}\,dx.$

Definition 3. [1, page 223] For integer k $\geq$ 0, $H^k(2\pi)$ is defined to be the closure of $C_p^k(2\pi)$ under the inner product norm $\|\varphi\|_{H^k} = \left[\sum_{j=0}^{k}\|\varphi^{(j)}\|_{L^2}^2\right]^{\frac{1}{2}}$. For arbitrary real s $\geq$ 0, H^s (2π) can also be obtained following [1, pages, 219-223]. For exact details of the Theorem 1 please see [1, page 223].

Theorem 1. [1, page 223] For $s \in R, H^s(2\pi)$ is the set of all series $\varphi(x) = \sum_{m=-\infty}^{\infty} a_m\psi(x)$ for which

$$\|\varphi\|_{*,s}^2 = |a_0|^2 + \sum_{|m|<0}|m|^{2s}|a_m|^2 < \infty$$

where

$\psi_m(x) = \frac{1}{\sqrt{2\pi}}e^{imx}, m = 0, \pm 1, \pm 2, \ldots$ Moreover, the norm $\|\varphi\|_{*,s}$ is equivalent to the standard Sobolev norm $\|\varphi\|_{H^s}$ for $\varphi \in$ Hs(2π).

Now from the DFT definition $\widetilde{U}_k = h\sum_{j=0}^{N-1} u(j\Delta x)e^{\frac{-i2\pi k x j}{2L}}$, and from the Fourier series definition $\hat{u}_k = \frac{1}{2L}\int_0^{2L} u(x)e^{\frac{i2\pi k x}{2L}}\,dx$ and the corresponding Fourier series is $u(x) = \sum_{k=-\infty}^{\infty}\hat{u}_k e^{\frac{-i2\pi k x}{2L}}$. Thus following [11] we have

$$\widetilde{U}_k = h\sum_{j=0}^{N-1}\sum_{s=-\infty}^{\infty}\hat{u}_s e^{\frac{i2\pi s x_j}{2L}} e^{\frac{-i2\pi k x_j}{2L}} = h\sum_{j=0}^{N-1}\sum_{s=-\infty}^{\infty}\hat{u}_s e^{\frac{i2\pi(s-k)x_j}{2L}}$$

$$= h\sum_{s=-\infty}^{\infty}\hat{u}_s\left[\sum_{j=0}^{N-1} e^{\frac{i2\pi\Delta x_j(s-k)}{2L}}\right] = h\sum_{s=-\infty}^{\infty}\hat{u}_s N\delta_N(s-k)$$

$$= \sum_{q=-\infty}^{\infty}\hat{u}_{k+qN}, \qquad (5)$$

Where $\delta_{(N)}(j) = \begin{cases} 1, & if \quad \text{j = qN for all q Z,} \\ 0, & otherwise. \end{cases}$

The relation (5) is known as discrete Poisson sum formula. Also from the Fourier series definition, the k^{th} Fourier coefficient can be written as

$$F_k = \frac{1}{2L}\int_0^{2L} f(x)e^{\frac{-i2\pi kx}{2L}}dx$$

$$= \frac{1}{2L}\int_0^{2L}\left(\sum_{j=-\infty}^{\infty}\left(f^{np}(x+2Lj)\right)\right)e^{\frac{-i2\pi kx}{2L}}dx$$

$$= \frac{1}{2L}\int_{-\infty}^{\infty}\left(f^{np}(y)e^{\frac{-i2\pi ky}{2L}}dy = \frac{1}{2L}\hat{f}^{np}\left(\frac{2\pi k}{2L}\right)\right.$$

which establishes the relation between Fourier co-efficient and the continuous Fourier transform. Using the Fourier transform definition (4) can be written as

$$\Delta x\sum_{k=0}^{N-1}\tilde{U}_k^{N+1}e^{\frac{2\pi ikx_j}{2L}} = \Delta x\sum_{k=0}^{N-1}\tilde{U}_k^{N}e^{\frac{2\pi ikx_j}{2L}} + \alpha\frac{\Delta t}{2\Delta x}\Delta x\sum_{k=0}^{N-1}\tilde{U}_k^{N}\left(e^{\frac{2\pi ikx_{j+1}}{2L}} - e^{\frac{2\pi ikx_{j-1}}{2L}}\right)$$

$$+\beta\frac{\Delta t}{\Delta x^2}\Delta x\sum_{k=0}^{N-1}\tilde{U}_k^{N}\left(e^{\frac{2\pi ikx_{j+1}}{2L}} - 2e^{\frac{2\pi ikx_j}{2L}}e^{\frac{2\pi ikx_{j-1}}{2L}}\right)$$

We simplify the above equation to get

$$\tilde{U}_k^{N+1} = \tilde{U}_k^{N}\left(1+\alpha\frac{\Delta t}{2\Delta x}\Delta x\left(e^{\frac{2\pi ik\Delta x}{2L}} - e^{\frac{-2\pi ik\Delta x}{2L}}\right) + \beta\frac{\Delta t}{\Delta x^2}\left(e^{\frac{2\pi ik\Delta x}{2L}} - 2 + e^{-\frac{2\pi ik\Delta x}{2L}}\right)\right)$$

and we write

$$\tilde{U}_k^{n+1} = g(\Delta x, \Delta t, k)\tilde{U}_k^{n},$$

Where

$$g(\Delta x,\Delta t,k) =$$

$$\left(1+\alpha\frac{\Delta t}{2\Delta x}\Delta x\left(e^{\frac{2\pi ik\Delta x}{2L}} - e^{\frac{-2\pi ik\Delta x}{2L}}\right) + \beta\frac{\Delta t}{\Delta x^2}\left(e^{\frac{2\pi ik\Delta x}{2L}} - 2 + e^{-\frac{2\pi ik\Delta x}{2L}}\right)\right)$$

$$= \left(1+2i\alpha\frac{\Delta t}{2\Delta x}\sin\left(\frac{2\pi k\Delta x}{2L}\right)\right) - 4\beta\frac{\Delta t}{\Delta x^2}\sin^2\left(\frac{\pi k\Delta x}{2L}\right)$$

Now repeated substitution of $\tilde{U}_k^n$, n = 0; 1; 2; _ _ _; in the above equation gives

$$\tilde{U}_k^n = g^n(\Delta x,\Delta t,k)\tilde{U}_k^0. \tag{6}$$

Thus the scheme (6) is stable [11] if

$$\left|g(\Delta x,\Delta t,k)\right| \le 1$$

Now

$$\left|g(\Delta x,\Delta t,k)\right|^2 = \left(1-4\beta\frac{\Delta t}{\Delta x^2}\sin^2\left(\frac{\pi k\Delta x^2}{2L}\right)\right)^2 + \left(\alpha\frac{\Delta t}{\Delta x}\sin\left(\frac{2\pi k\Delta x}{2L}\right)\right)^2. \tag{7}$$

Here $\left|g(\Delta x,\Delta t,k)\right|^2 \le 1$, gives

$$\Delta t \le \frac{2\beta\Delta x^2}{4\beta^2+\alpha^2\Delta x}.$$

Thus we conclude with the following stability result.

Theorem 1. The one step scheme (4) for the IBVP is stable if $0 < \Delta t \le \dfrac{2\beta\Delta x^2}{4\beta^2+\alpha^2\Delta x}$.

IV. Accuracy and Convergence Analysis

In this section we analyze convergence of the one step approximation (4) used to solve (2). This analysis is based on [11]. We define the exact solution of (2) as

$$u(x,t) = \sum_{j=-\infty}^{\infty} B_j(t)e^{\frac{2ij\pi x}{2L}}. \tag{8}$$

Differentiating (8) with respect to x and t we get

$$u_x(x,t) = \sum_{j=-\infty}^{\infty}\frac{2ij\pi}{2L}B_j(t)e^{\frac{2ij\pi x}{2L}}, \tag{9}$$

$$u_{xx}(x,t) = \sum_{j=-\infty}^{\infty}\left(\frac{2ij\pi}{2L}\right)^2 B_j(t)e^{\frac{2ij\pi x}{2L}}, \tag{10}$$

and

$$u_{xx}(x,t) = \sum_{j=-\infty}^{\infty}\left(\frac{2ij\pi}{2L}\right)^2 B_j(t)e^{\frac{2ij\pi x}{2L}}, \tag{11}$$

and $u_t(x,t) = \sum_{j=-\infty}^{\infty} B'_j(t)e^{\frac{2ij\pi x}{2L}}$.

Substituting (9), (10) and (11) in (2) and then simplifying we write

$$B'_j(t) = \left(\alpha\frac{2ij\pi}{2L} - \beta\frac{4j^2\pi^2}{4L^2}\right)B_j(t)$$

$$\Rightarrow B'_j(t) = qB_j(t) \tag{12}$$

where

$$q = \left(\alpha\frac{ij\pi}{L} - \beta\frac{j^2\pi^2}{L^2}\right).$$

The solution of (12) can be written as

$$B_j(t) = B_j(0)e^{qt}, \quad (13)$$

where

$$B_j(0) = \frac{1}{2L}\int_{-L}^{L} u(x,0)e^{\frac{-ij\pi x}{L}}dx$$

Now using (13) the exact solution of (2) can be written as

$$u(x,t) = \sum_{j=-\infty}^{\infty} B_j(0)e^{qt}e^{\frac{ij\pi x}{L}}, \quad (14)$$

Definition 4. (similar to [11]) An initial value problem of type (12) is well-posed if for any choice of initial solution Bj(0), there exists a constant C such that the following inequality holds

$$\|B_j(t)\| \le C(t)\|B_j(0)\|.$$

From the above definition it can be verified that the differential equation of type (12) is well-posed if and only if there exists a constant c[1] such that

$$\mathrm{Re}(q) \le c'. \quad (15)$$

Then we return to our main discussion. Applying the inverse of the DFT in (6)

$$U_j^n = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} g^n \widetilde{U}_k^0 e^{\frac{ik\pi x_j}{L}}. \quad (16)$$

Thus from (14) and (16)

$$u(x_j, t_m) - U_j^m$$

$$= \sum_{n=-\infty}^{\infty} B_n(0)e^{qt_m}e^{\frac{ij\pi x_j}{L}} - \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} g^m \widetilde{U}_k^0 e^{\frac{ik\pi x_j}{L}}$$

$$= \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \left(B_k(0)e^{qt_m} - g^m \widetilde{U}_k^0\right)e^{\frac{i\pi k x_j}{L}} + \sum_{|k|>\frac{N}{2}} B_k(0)e^{qt_m}e^{\frac{ik\pi x_j}{L}}.$$

So

$$\|u(x_j, t_m) - U_j^m\|_h^2 \le \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \left\|\left(B_k(0)e^{qt_m} - g^m \widetilde{U}_k^0\right)\right\|^2 +$$

$$\sum_{|k|>\frac{N}{2}} \left\|B_k(0)e^{qt_m}e^{\frac{ik\pi x_j}{L}}\right\|^2. \quad (17)$$

To get bound and to distinguish between periodic and non-periodic functions (u(x; t)) here by u^{np} we mean a function u^{np} (x, t) ∈ $\mathrm{R} \times R^+$ and u(x, t) ∈ [0, 2L] $\times R^+$. Using the Poisson summation formula

$$B_k(0) = \frac{1}{2L}\hat{u}_k^{np}(0)$$

where u^{np} represents u(x, t) in infinite domain, see [11] for similar discussion.

Now

$$\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \left\|B_k(0)e^{qt_m} - g^m \widetilde{U}_k^0\right\|^2 \le \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \left\|\left(e^{qt_m} - g^m\right)\widetilde{U}_k^{np,0}\right\|^2$$

$$+ \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \left\|g^m \sum_{s\neq 0} \hat{u}^{np,0}\left(\frac{2\pi(k+sN)}{2L}\right)\right\|^2$$

and

$$e^{q\Delta t} - g(\Delta t, \Delta x, k)$$

$$= e^{\Delta t\left(\frac{\alpha i k\pi}{L} - \frac{Bk^2\pi^2}{L^2}\right)} - \left(1 + \alpha\frac{\Delta t}{2\Delta x}2i\sin\left(\frac{\pi k\Delta x}{L}\right) - 4\beta\frac{\Delta t}{\Delta x^2}\sin^2\left(\frac{\pi k\Delta x}{2L}\right)\right)$$

$$= \frac{1}{2}\left(\frac{i\alpha k\pi}{L} - \frac{\beta k^2\pi^2}{L^2}\right)\Delta t^2 + i\alpha\Delta t\Delta x^2\left(\frac{\pi k}{L}\right)^3 + \mathrm{O}\left(\Delta \mathrm{x}^4\right)$$

$$+ \beta\Delta t\left(\frac{\pi^4 k^4}{2L^4}\Delta x^2 + \mathrm{O}\left(\Delta \mathrm{x}^4\right)\right) + O\left(\Delta \mathrm{t}^3\right)$$

$$= \Delta t\left(c_1\Delta t + c_2\Delta x^2\right) + \mathrm{O}\left(\Delta \mathrm{t}^3, \Delta \mathrm{x}^4\right)$$

where c_1, $c_2 \in C$. Thus

$$\left\|e^{q\Delta t} - g(\Delta t, \Delta x, k)\right\| \le \Delta t\left(c_1\Delta t + c_2\Delta x^2\right) \le$$

$$\Delta t\left(c_1^*\Delta t + c_2^*\Delta x^2\right), \quad (18)$$

where $C_1^*, C_2^* \in R$ gives the accuracy of the scheme. So,

$$\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \left\| \left(e^{qt_n} - g^n(\Delta t, dx, k) \right) \tilde{u}_k^{np,0} \right\|^2 = \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \left\| \left(\left(e^{q\Delta t}\right)^n - g^n(\Delta t, dx, k) \right) \tilde{u}_0^{np} \right\|^2$$

$$= \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \left\| n\left(e^{q\Delta t} - g(\Delta t, dx, k) \right) \tilde{u}_0^{np} \right\|^2$$

if $\left| e^{q\Delta t} \right| \leq 1$ and $|g| \leq 1$ hold

$$\leq \Delta t^2 n^2 \left(C_1^* \Delta t + C_2^* \Delta x^2 \right)^2 \left\| u_0 \right\|_{H^0(2L)}^2 = t^2 \left(C_1^* \Delta t + C_2^* \Delta x^2 \right)^2 \left\| u_0 \right\|_{H^0(2L)}^2 . \quad (19)$$

$$\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \left\| g^n \sum_{s\neq 0} \hat{u}_0^{np} \left(\frac{2\pi(k+sN)}{2L} \right) \right\|^2 \leq \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \left\| \sum_{s\neq 0} \hat{u}_0^{np} \left(\frac{2\pi(k+sN)}{2L} \right) \right\|^2$$

$$\leq 2L \sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \left\| \sum_{s\neq 0} \hat{u}_0 \left(\frac{2\pi(k+sN)}{2L} \right) \right\|^2 \leq 2LC(v)\Delta x^{2v} \left\| u_0 \right\|_{,v}$$

$$= C(v)\Delta x^{2v} \left\| u_0 \right\|_{H^v}(2L) \quad (20)$$

for some $v \geq 1/2$ and

$$\sum_{|k|>\frac{N}{2}} \left| B_k(0) e^{qt_n} \right|^2 \leq \sum_{|k|>\frac{N}{2}} \left| B_k(0) \right|^2$$

$$= \frac{1}{N^{2v}} \sum_{|k|>\frac{N}{2}} N^{2v} \left| B_k(0) \right|^2 \leq C(v)\Delta x^{2v} \left\| u_0 \right\|_{H(2L)^v} \quad (21)$$

Thus combining (19), (20) and (21), the error estimate (17) can be written as

$$\left\| u(x_j, t_n) - U_j^n \right\|^2 \leq t^2 \left(c_1^* \Delta t + c_2^* \Delta x^2 \right)^2 \left\| u_0 \right\|_{H^0(2L)}^2 + C(v)\Delta x^{2v} \left\| u_0 \right\|_{H^u(2L)}$$

Thus we conclude the convergence estimate of the scheme used to solve (2) with some smooth initial function by the following theorem.

Theorem 2. If the initial boundary value problem (2) with some smooth initial function u_0(x, 0) = g(x) is approximated by the stable one step finite difference formula (4), then there exists nonnegative constants C_1^*, C_2^* and C such that the following inequality holds

$$\left\| u(x_j, t_n) - U_j^n \right\| \leq t\left(C_1^* \Delta t + C_2^* \Delta x^2 \right) \left\| u_0 \right\|_{H^0(2L)} + C(v)\Delta x^v \left\| u_0 \right\|_{H^v(2L)}$$

where $v > \frac{1}{2}$.

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