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Solution of Transcendental Equation Using Clamped Cubic Spline

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Abstract

Bisection and regular false-position methods are widely used to find roots of a transcendental function *f(x)* in a certain interval [*a,b*] satisfying *f(a). f(b) < 0*. The paper develops a new algorithm to find roots of transcendental functions based on false position method. We use two end points of the interval to interpolate *f(x)* by an equivalent cubic polynomial using clamped cubic spline formula. We consider one of the roots of the interpolated function to define a new interval and to approximate the root of *f(x)*.

Keywords: Transcendental equation, regular false-position method, interpolation, clamped cubic spline formula

I. Introduction

Transcendental equations are equations containing terms that are trigonometric, algebraic, exponential, logarithmic, etc. terms. Many analytical and iterative methods are used to solve transcendental equations e.g. [2],[5]. Though these methods are capable of solving many transcendental equations they suffer from many common disadvantages. Usually transcendental equations have many solutions in a given range, and analytical methods are not able to find all these roots in a given interval. Even when they find several solutions, it is not possible to conclude that the given method has found the complete set of roots/solutions, and has not missed any particular solution. Also, these methods fail in case of misbehaved or discontinuous functions. Hence, though these methods may work very well in some situations, they are not general in nature and need a lot of homework from the Analyst.

Newton Raphson method is a commonly used method for solving transcendental equations. The method makes use of the slope of the curve at different points. Therefore, if the function is non differentiable at points or has a point of inflection, the method is not able to find the root. Secondly, if the function changes its slope very quickly (frequently achieves slope of zero), or is discontinuous, it cannot be solved by this method. If the function is discrete, the derivative has no meaning for it and this method cannot be used. Also there is no straightforward way to find all the roots in an interval or even ascertain the number of roots in the interval.

Bisection method needs two points $(a, f(a))$ and $(b, f(b))$ on the graph such that $f(a)\cdot f(b)<0$. There is no straightforward analytical method to find these points. Another problem lies in choosing the distance between the points a and b. For the method to work, a and b should be close enough, such that the function behaves monotonously in these limits. At the same time, a small difference in values of a and b makes it difficult to search the sample space. An algorithm to ascertain such points for all roots of the equation has to be essentially random in nature and it can be another application of genetic algorithm (GA). This will be discussed later. Further still, the method fails for discontinuities in a function.

Method of False Position suffers from the same problems as in Bisection method. Hence it can be concluded that analytical methods cannot find all the roots of a transcendental equation reliably [1].

To develop a new algorithm of finding roots of a transcendental equation, we assume $f(x)=0$ in an interval $[a,b]$ so that $f(a)$ and $f(b)$ are of opposite signs. So, $f(x)$ cuts the x-axis at one point in the interval $[a,b]$ at least which is guaranteed by the following theorems.

Theorem 1: If $f(x)$ is a real and continuous function in an interval $a\le x\le b$, and $f(a)$ and $f(b)$ are of opposite signs [2] i.e. $f(a)\cdot f(b)<0$ then there is at least one real root in the interval between a and b.

By clamped cubic spline interpolation we can easily make a cubic polynomial using two end points of the function $(a, f(a))$ and $(b, f(b))$ that can represent our assumed transcendental function. How a cubic equation can be solved by a general formula and how the clamped cubic spline formula makes a cubic polynomial are described in section II. From a cubic equation, out of the three roots we must find at least one real root x_1 lying in the interval $[a,b]$ which is our first approximate root of $f(x)=0$. Checking sign of $f(x_1)$ we replace x_1 for a or b for furthers iterations. The general formulation of Numerical solutions of transcendental equations based on cubic polynomials is described in section III and in section IV we take examples to illustrate the working rule and its geometric interpretation.

II. Preliminary Results

In the early 16th century the Italian mathematician Scipione del Ferro (1465-1526) found a method for solving a class of cubic equations of the form $x^3+mx=n$. In 1530 there was a famous contest between Niccolò Tartaglia (1500–1557) and Antonio Fiore, the student of Del Ferro,

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where Tartaglia won the contest. Gerolamo Cardano (1501–1576) learned about Ferro's prior work and published Ferro's method in his book *Ars Magna* in 1545. Again Tartaglia challenged to Cardano, which Cardano denied. The challenge was eventually accepted by Cardano's student Lodovico Ferrari (1522–1565). Ferrari did better than Tartaglia in the competition, and Tartaglia lost both his prestige and income [3].

The general cubic equation has the form

$$ax^3 + bx^2 + cx + d = 0 \quad (1)$$

with $a \neq 0$ and the coefficients a, b, c, d are generally assumed to be real numbers.

We can distinguish several possible cases using the discriminant,

$$\Delta = 18abcd - 4b^3d + b^2c^2 - 4ac^3 - 27a^2d^2. \quad (2)$$

The following cases need to be considered: [4]

- If $\Delta > 0$, then the equation has three distinct real roots.
- If $\Delta = 0$, then the equation has a multiple root and all its roots are real.
- If $\Delta < 0$, then the equation has one real root and two nonreal complex conjugate roots.

Every cubic equation (1) with real coefficients has at least one solution x among the real numbers; this is a consequence of the intermediate value theorem. And the real solution is:

$$x = -\frac{b}{3a} - \frac{1}{3a}\sqrt[3]{\frac{1}{2}\left[2b^3 - 9abc + 27a^2d + \sqrt{(2b^3 - 9abc + 27a^2d)^2 - 4(b^2 - 3ac)^3}\right]} - \frac{1}{3a}\sqrt[3]{\frac{1}{2}\left[2b^3 - 9abc + 27a^2d - \sqrt{(2b^3 - 9abc + 27a^2d)^2 - 4(b^2 - 3ac)^3}\right]} \quad (3)$$

Clamped cubic spline interpolation

The most common piecewise-polynomial approximation uses cubic polynomials between each successive pair of nodes and is called cubic spline interpolation**.** A general cubic polynomial involves four constants, so there is sufficient flexibility in the cubic spline procedure to ensure that the interpolant is not only continuously differentiable on the interval, but also has a continuous second derivative. The construction of the cubic spline does not, however, assume that the derivatives of the interpolant agree with those of the function it is approximating, even at the nodes shown in fig. 1.

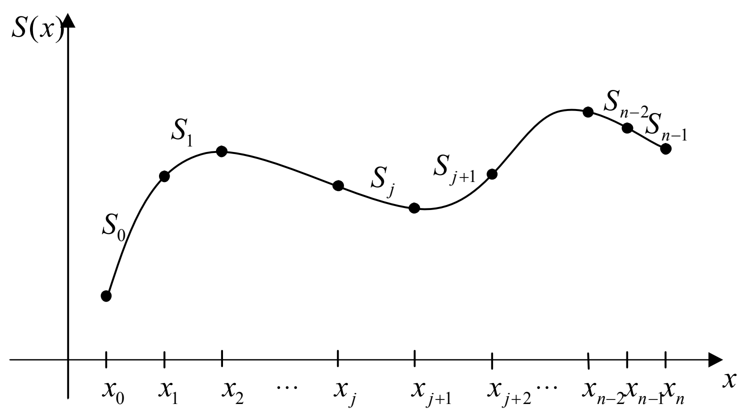


Fig. 1. Interpolant of a curve in different subintervals

Given a function f defined on $[a,b]$ and a set of nodes $a = x_0 < x_1 < \cdots < x_n = b$, a clamped cubic spline interpolant S for f is a function that satisfies the following conditions: [5]

a. $S(x)$ is a cubic polynomial, denoted $S_j(x)$, on the subinterval $[x_j, x_{j+1}]$ for each $j = 0, 1, \cdots, n-1$;

b. $S(x_j) = f(x_j)$ for each $j = 0, 1, \cdots, n$;

c. $S_{j+1}(x_{j+1}) = S_j(x_{j+1})$ for each $j = 0, 1, \cdots, n-2$;

d. $S'_{j+1}(x_{j+1}) = S'_j(x_{j+1})$ for each $j = 0, 1, \cdots, n-2$;

e. $S''_{j+1}(x_{j+1}) = S''_j(x_{j+1})$ for each $j = 0, 1, \cdots, n-2$;

f. $S'(x_0) = f'(x_0)$ and $S'(x_n) = f'(x_n)$ (clamped boundary).

To construct the clamped cubic spline interpolant for a given function f, the conditions in the definition are applied to the cubic polynomials

$$S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3, \tag{4}$$

for each $j = 0, 1, \cdots, n-1$.

Theorem 2: If f is defined at $a = x_0 < x_1 < \cdots < x_n = b$ and differentiable at a and b, then f has a unique clamped spline interpolant S on nodes $x_0, x_1, \cdots, x_n$; that is, a spline interpolant that satisfies the boundary conditions $S'(a) = f'(a)$ and $S'(b) = f'(b)$ [5].

III. Numerical Solutions of Transcendental Equations Based on Cubic Polynomials

Given a function f defined on $[a,b]$ and a set of nodes $a = x_0 < x_1 = b$. In this case $n = 1$, a cubic spline interpolant S for f is to be found that satisfies the conditions of clamped cubic spline interpolation formulas. Since $n = 1$, conditions (a), (b) and (f) are applicable only. Here we have to find a cubic polynomial

$$S(x) = a_0 + b_0(x - x_0) + c_0(x - x_0)^2 + d_0(x - x_0)^3 \tag{5}$$

very close to a function f defined on $[a,b]$. We get

$$S(x_0) = a_0 = f(x_0) \tag{6}$$

and

$$S'(x_0) = b_0 = f'(x_0). \tag{7}$$

Now,

$$S(x_1) = a_0 + b_0(x_1 - x_0) + c_0(x_1 - x_0)^2 + d_0(x_1 - x_0)^3 = f(x_1) \tag{8}$$

Using the results obtained from (6) and (7), equation (8) becomes

$$c_0(x_1 - x_0) + d_0(x_1 - x_0)^2 = \frac{f(x_1) - f(x_0)}{x_1 - x_0} - f'(x_0). \tag{9}$$

Again,

$$S'(x_1) = b_0 + 2c_0(x_1 - x_0) + 3d_0(x_1 - x_0)^2 = f'(x_1). \tag{10}$$

Using the results of (6) and (7), equation (10) becomes

$$2c_0(x_1 - x_0) + 3d_0(x_1 - x_0)^2 = f'(x_1) - f'(x_0). \tag{11}$$

Solving (9) and (11) we get,

$$c_0 = \frac{3[f(x_1) - f(x_0)]}{(x_1 - x_0)^2} - \frac{f'(x_1) + 2f'(x_0)}{x_1 - x_0} \tag{12}$$

and

$$d_0 = \frac{f'(x_1) + f'(x_0)}{(x_1 - x_0)^2} - \frac{2[f(x_1) - f(x_0)]}{(x_1 - x_0)^3}. \tag{13}$$

The following algorithm is used for computer implementation of the scheme.

Algorithm: To find a solution of $f(x) = 0$ given the continuous function f on the interval $[\alpha, \beta]$ where $f(\alpha)$ and $f(\beta)$ have opposite signs:

INPUT endpoints α, β; tolerance TOL; maximum number of iterations m

OUTPUT approximate solution x_0 or message of failure

Step 1 Set $i = 1$

Step 2 set $x_0 = \alpha$ and $x_1 = \beta$

Step 3 while $i \le \mathrm{m}$ do Step 4 to Step 12

Step 4 set $a_0 = f(x_0)$ and $b_0 = f'(x_0)$

Step 5 set

$$c_0 = \frac{3[f(x_1) - f(x_0)]}{(x_1 - x_0)^2} - \frac{f'(x_1) + 2f'(x_0)}{x_1 - x_0}$$ and

$$d_0 = \frac{f'(x_1) + f'(x_0)}{(x_1 - x_0)^2} - \frac{2[f(x_1) - f(x_0)]}{(x_1 - x_0)^3}$$

Step 6 set

$$S(x) = a_0 + b_0(x - x_0) + c_0(x - x_0)^2 + d_0(x - x_0)^3$$

Step 7 rearrange step 6 of the form $S(x) = ax^3 + bx^2 + cx + d$

Step 8 set

$$p = -\frac{b}{3a} - \frac{1}{3a}\sqrt[3]{\frac{1}{2}\left[2b^3 - 9abc + 27a^2 d + \sqrt{(2b^3 - 9abc + 27a^2 d)^2 - 4(b^2 - 3ac)^3}\right]} - \frac{1}{3a}\sqrt[3]{\frac{1}{2}\left[2b^3 - 9abc + 27a^2 d - \sqrt{(2b^3 - 9abc + 27a^2 d)^2 - 4(b^2 - 3ac)^3}\right]}$$

Step 10 If $f(\alpha) f(p) > 0$ then set $\alpha = p$

else set $\beta = p$

Step 11 If $|f(p)| < TOL$ then

OUTPUT (p) (Procedure completed successfully)

STOP

Step 12 Set $i = i + 1$

Step 13 OUTPUT (Method failed after m iterations)

(Procedure completed unsuccessfully)

STOP

IV. Numerical results and geometric interpretation

In this section we experiment our scheme to solve transcendental functions and compare our results with exact roots of the functions.

Example 1: To find a root of the function [5]

$$f(x) = e^{x} + 2^{-x} + \cos x - 6 \tag{14}$$

Using our present method we need two initial approximations a and b so that $f(a) < 0$ and $f(b) > 0$.

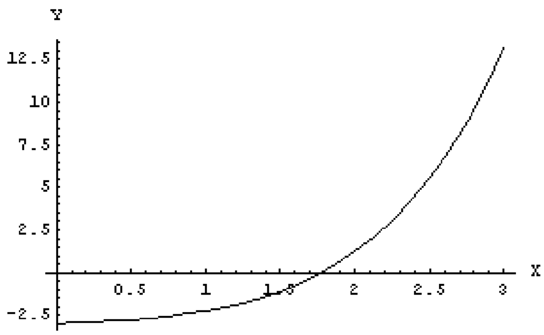


Fig. 2. Graph of $f(x)$

Since $f(1) = -2.24142 < 0$ and $f(3) = 13.2205 > 0$ so, we choose $a = 1, b = 3$. Now using formula (6), (7), (12) and (13) we can find the simplified form of (5)

$$S(x) = 1.48151x^{3} - 4.30719x^{2} + 5.70008x - 5.11582 \tag{15}$$

To find a real root of the above equation we use the formula (3) and find $x_1 = 1.83611$, which is an approximate root of $f(x)$ in the first iteration.

Since $f(x_1) = f(1.83611) = 0.28995 > 0$,

we replace the value $x_1 = 1.83611$ instead of the value of b. Again we have to find out a cubic equation and its real root. Continuing this procedure we can find a root of the transcendental equation (14).

Table. 1. Comparison of the result generating by present method with bisection method and false position method in [1, 3] with tolerance 0.0001.

No. of iteration	Regular false-position method	Bisection method	Present method	Exact root using *Mathematica*
1	1.2899264806410	2.0000000000000	1.8361095176046	1.7771822744895
2	1.4828672009263	1.5000000000000	1.7772680527096	
3	1.6041635463171	1.7500000000000	1.7771822746684	
4	1.6772743547939	1.8750000000000		
5	1.7201273029210	1.8125000000000		
6	1.7448136944489	1.7812500000000		
7	1.7588890365717	1.7656250000000		
8	1.7668664291911	1.7734375000000		
9	1.7713722534192	1.7773437500000		
10	1.7739122998661	1.7753906250000		
11	1.7753426113893	1.7763671875000		
12	1.7761475257975	1.7768554687500		
13	1.7766003365261	1.7770996093750		
14	1.7768550184506	1.7772216796875		
15	1.7769982475894	1.7771606445313		
16	1.7770787923973	1.7771911621094		
17	1.7771240851260			
18	1.7771495540650			
19	1.7771638755616			

Taking two end points $a = 1$ and $b = 3$ with tolerance 0.0001 in the three cases we observe that the false position method takes 19 iterations the bisection method takes 16 iterations and our present method takes only 3 iterations to reach the result with desired tolerance.

Example 2: To find a root of the function [6]

$$f(x) = e^{x} - x - 2, \qquad 1 < x < 2. \tag{16}$$

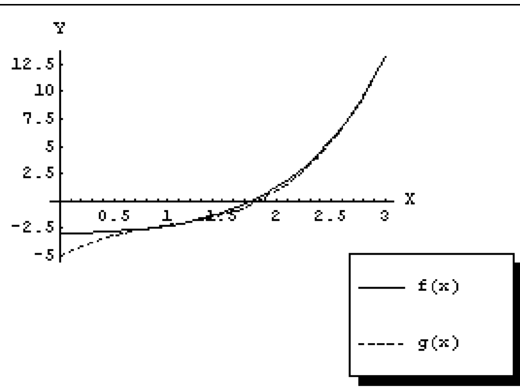
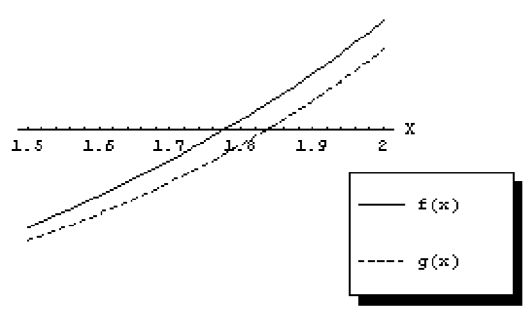
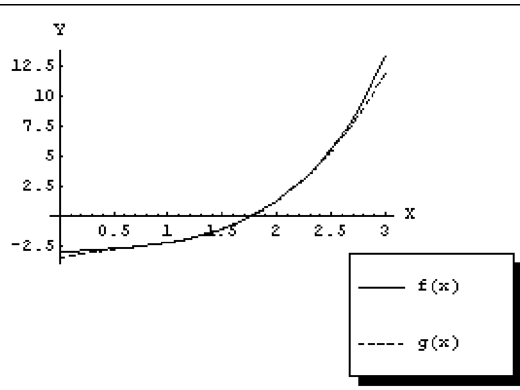
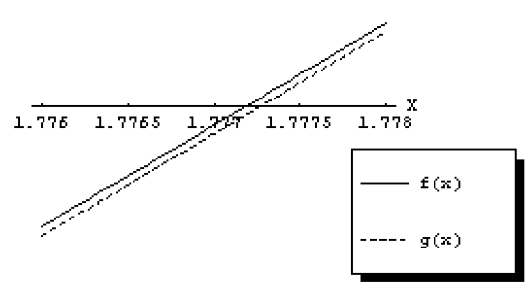
Table. 2. Comparison of the result generating by present method with bisection method and false position method in [1, 2] with tolerance 0.0001.

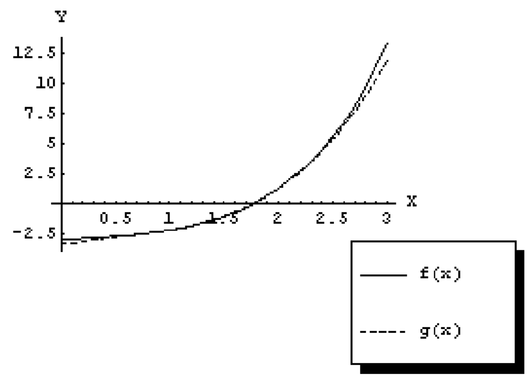
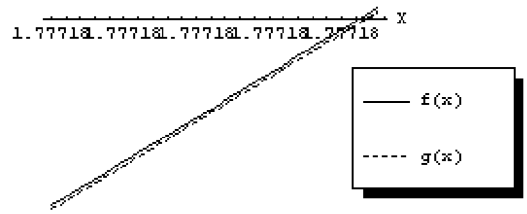
No. of iteration	Regular false-position method	Bisection method	Present method	Exact root using *Mathematica*
1	1.0767462531835	1.5000000000000	1.1474980831453	1.1461932206206
2	1.1137822648029	1.2500000000000	1.1461932227191	
3	1.1311953424248	1.1250000000000	1.1461932206206	
4	1.1392808032624	1.1875000000000		
5	1.1430132465703	1.1562500000000		
6	1.1447315652384	1.1406250000000		
7	1.1455216447739	1.1484375000000		
8	1.1458847126630	1.1445312500000		
9	1.1460515103022	1.1464843750000		
10	1.1461281297700	1.1455078125000		
11		1.1459960937500		
12		1.1462402343750		
13		1.1461181640625		
14		1.1461791992188		

IV. a. Geometric Interpretation

Geometrically, a cubic polynomial passing through two points like $(a, f(a))$ and $(b, f(b))$ is a twisting curve. In this case, the cubic polynomial found by clamped cubic spline is a curve for which initial and end points in a fixed interval as well as slopes at those points are similar to that of $f(x)$ respectively and cuts the x-axis closed to a zero of $f(x)$. Using the intersecting point we can repeat the process again to obtain a better result and so on. The figures in Table 3 show the iterations of cubic approximation of $f(x)$.

Table. 3. Geometric interpretation of the iterations of the present method

No. of iteration	Graph of $f(x)$ with cubic approximation: In remote view	Graph of $f(x)$ with cubic approximation: In close view
1		
2		

3		

V. Conclusion

Regular false position method uses straight lines to generate an approximate root of any function $f(x)$. In the present method we use cubic polynomials (twisted curves) so that the root converges faster, which is shown the Table 3. So, this method provides faster approximation than the regular false-position and bisection method for transcendental equations.

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