A General Construction Method of Simultaneous Confounding in $p^n$ - Factorial Experiment

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Abstract

A general construction method of simultaneous confounding in $p^n$ ($p$ is prime) factorial experiment is proposed. The concept of matrix method in the construction of factorial experiment with a single factorial effect confounded is used to develop the method. The procedure may be extended for the construction of simultaneous confounding of factorial experiment with three or more factorial effects confounded.

I. Introduction

The researchers working with factorial experiments experience difficulty especially when the number of factors as well as the number of levels of each factor is large. It becomes more difficult if we have no required number of homogeneous plots in practice. In such situations, we are bound to use a limited number of homogeneous plots to analyze the factorial effects. As a result, some factorial effects or interactions will be mixed up with block effect, i.e. confounded. Since there is no way to avoid this, the higher order interaction effects are usually considered to be confounded.

Bose and Kishan (1940), Bose (1947) described the construction of $p^n$ factorial designs using finite geometries. The treatments are represented by $n$-tuples $(a_1, \ldots, a_n)$ where $a_i$ are elements of $GF(p)$. The method is available only when $p$ is prime or prime power. A system of simultaneous confounding in $2^n$ factorial experiment has been described, where an intrablock subgroup is constructed with the common elements taken from the factorial effects of two incomplete blocks, each confounded with a single factorial effect (Kempthorne, 1947, 1952). Das (1964) described an equivalent method of Bose in which some of the treatment factors are designated as basic factors and the others as added factors. Levels of added factors are derived by combination of the levels of the basic factors over $GF(p)$. White and Hultquist (1965) extended the field method to design with number of levels of treatment factors. John and Dean (1975) described the construction of a particular class of single replicate block designs, which they call generalized cyclic designs. The essential feature of the method is that the $n$-tuples giving the treatments of a particular block constitute an Abelian group, the intrablock subgroup. Patterson (1976) described a general computer algorithm, called DSIGN, in which levels of treatment factors are derived by linear combinations of levels of plot and block factors. The method provides finite-field, generalized cyclic and other designs. Mallick, S. A. (1973 & 1975) developed two systems of designing factorial effects with simultaneous confounding of two effects, one for $3^n$ and the other for $4^n$ - factorial experiments. In these systems of simultaneous confounding, the combination of levels was based on some manipulating manner. Jalil, et. al. (1990) developed a matrix method of designing a single factorial effect confounded in a $p^n$ - factorial experiment, where the level combinations are obtained by matrix operations of the levels. Construction method of simultaneous confounding has been developed independently for $3^n$ and $5^n$ factorial experiments (Jalil and Mallick, 2010). The present work is a general method of construction for simultaneous confounding in a $p^n$ - ($p$ is prime) factorial experiment.

II. Notation and Definition

The formulae given below have been used in determining the number of incomplete blocks or intrablocks ($b$) and the number of homogeneous plots in each incomplete block or intrablock ($k$) for a single or simultaneously confounded factorial experiment (symmetrical).

Let $b$ be the number of incomplete blocks or intrablocks in a confounded experiment; and $k$ be the number of plots in each incomplete block or intrablock. Then,

$$b = \frac{p^n}{p^{n-r}} \quad \text{and} \quad k = \frac{p^n}{p^r};$$

where $r$ is the number of factorial effects to be confounded.

For a single factorial effect confounding in $p^n$ factorial experiment, the number of incomplete blocks is computed as,

$$b = \frac{p^n}{p^{n-1}} = p$$

and the number of combinations of levels in each incomplete block is given by,

$$k = \frac{p^n}{p} = p^{n-1}.$$  

For a simultaneous confounding (two factorial effects) in $p^n$ factorial experiment, the number of intrablock subgroups is computed as,

$$b = \frac{p^n}{p^{n-2}} = p^2$$

and the number of level
combinations in each intrablock subgroup is given by,
\[ k = \frac{p^n}{p^2} = p^{n-2}. \]

The general construction matrix with its intrablock subgroups of simultaneous confounding in a \( p^n \) factorial experiment can be represented as,

\[
\begin{pmatrix}
B_1 & B_{p+1} & \cdots & B_{(p-1)p+1} \\
B_2 & B_{p+2} & \cdots & B_{(p-1)p+2} \\
\vdots & \vdots & \ddots & \vdots \\
B_p & B_{2p} & \cdots & B_{p^2}
\end{pmatrix} \quad \text{and}
\]

in particular, the construction matrices for \( 3^n \) and \( 5^n \) factorial experiments are given respectively by,

\[
\begin{pmatrix}
B_1 & B_4 & B_7 \\
B_2 & B_5 & B_8 \\
B_3 & B_6 & B_9
\end{pmatrix} \quad \text{and}
\begin{pmatrix}
\text{Col.1} & \text{Col.2} & \text{Col.3} & \text{Col.4} & \text{Col.5} \\
B_1 & B_2 & B_3 & B_4 & B_5 \\
B_6 & B_7 & B_8 & B_9 & B_{10} \\
B_{11} & B_{12} & B_{13} & B_{14} & B_{15} \\
B_{16} & B_{17} & B_{18} & B_{19} & B_{20} \\
B_{21} & B_{22} & B_{23} & B_{24} & B_{25}
\end{pmatrix}
\]

III. Construction Method of Simultaneous Confounding in \( p^n \) Factorial Experiments

In the construction of \( p^n \) (\( p \) is prime) factorial experiment with a single factorial effect confounded, we can write the level combinations by the matrix method described below (Jalil, et al., 1990).

\[ M = [M_0, M_1, M_2, \cdots, M_{p-1}]; \quad (1) \]

where incomplete blocks \( M_a; a = 0, 1, 2, \cdots, (p-1) \) is given by:

\[ M_a = [V_1(p^0), V_2(p^1), \cdots, V_u(p^{p-1}), \cdots, a_p]_{p^{n-1} \times n}, \]

with

\[ V_i(p^{p-1}) = p^i \times \{0I_{p^{(n-i-1)}}, 0I_{p^{(n-2-i)}}, \cdots, (p-1)I_{p^{(n-1-i)}}\}, \]

each is a column vector of dimension \( p^{n-1} \).

\( (p^j) = p^j \times \text{times repetitions of the elements of } V_j \) in ascending ordered level;

\[ u = 0, 1, 2, \cdots, (p-1); \quad i = 1, 2, 3, \cdots, (n-1); \]

\( j = 0, 1, 2, \cdots, (p-1); \) with the restriction that \( i = j + 1 \);

\( I_m \); sum vector of dimension \( m \); and

\( a_p = [a_{u1}, a_{u2}, \cdots, a_{up}]^T \) is called the adjustment vector.

From the equation (1), \( M_0 \) is called the key incomplete block of a single factorial effect confounded in a \( p^n \) factorial experiment. For a plan of simultaneous confounding of two factorial effects in a \( p^n \)-factorial experiment, we are to perform the following steps.

Step 1. Find the independent key incomplete blocks for the factorial effects to be confounded simultaneously using Equation 1. Let these key incomplete blocks be denoted by, \( M_0 \), which represents the level combinations of the key incomplete block for the first confounded factorial effect and \( M_0 \) represents the level combinations of the key incomplete block for the second confounded factorial effect.

Step 2. Find the common elements of level combinations (row vectors) of these two key incomplete blocks and form a matrix, can be denoted by \( B_1 \). \( B_1 \) is called the key intrablock subgroup of level combinations of two factorial effects confounded simultaneously. It can be seen that the key intrablock subgroup contains the combination of the lowest levels for all the factors.

Step 3. To find all other intrablock subgroups of the construction matrix we will follow the computations procedures described below.

The intrablock subgroups \( B_2, B_3, \cdots, B_P \), below the key block \( B_1 \) are obtained by adding the vectors \((\cdots 0 \ 1 \ 0 \ \cdots)\); \( c = 1, 2, \cdots, p \) with each of the elements (row vectors) of the key intrablock subgroup \( B_1 \) as described below.

\( B_2 \) is obtained by an addition of the vector \((\cdots 0 \ 1 \ 0 \ \cdots)\) with each of the vector elements of \( B_1 \);

\( B_3 \) is obtained by an addition of the vector \((\cdots 0 \ 2 \ 0 \ \cdots)\) with each of the vector elements of \( B_1 \);

\[ \cdots \ldots \ldots \]

\( B_P \) is obtained by an addition of the vector \((\cdots 0 \ p \ 0 \ \cdots)\) with each of the vector elements of \( B_1 \).

\( B_1, B_2, B_3, \cdots, B_P \) are the intrablock subgroups placed in the first column of the construction matrix.

After having the intrablock subgroups in the first column we will get the intrablock subgroups \( B_{p+1}, B_{p+2}, \cdots, B_{2p} \) of the second column, which can be computed as shown below.

\( B_{p+1} \) is obtained by an addition of the vector \((\cdots 0 \ 0 \ 1 \ \cdots)\) with each of the vector elements of \( B_1 \);

\( B_{p+2} \) is obtained by an addition of the vector \((\cdots 0 \ 0 \ 1 \ \cdots)\) with each of the vector elements of \( B_2 \);
$B_{p+3}$ is obtained by an addition of the vector $(-0001)$ with each of the vector elements of $B_3$;

\[ \ldots \ldots \ldots \]

$B_{2p}$ is obtained by an addition of the vector $(-0001)$ with each of the vector elements of $B_p$.

The intrablock subgroups $B_{2p+1}$, $B_{2p+2}$, \ldots, $B_{1p}$ placed in the third column can be computed as shown below.

$B_{2p+1}$ is obtained by an addition of the vector $(-0002)$ with each of the vector elements of $B_1$;

$B_{2p+2}$ is obtained by an addition of the vector $(-0002)$ with each of the vector elements of $B_2$;

$B_{2p+3}$ is obtained by an addition of the vector $(-0002)$ with each of the vector elements of $B_1$;

\[ \ldots \ldots \ldots \]

$B_{1p}$ is obtained by an addition of the vector $(-0002)$ with each of the vector elements of $B_p$.

\[ \ldots \ldots \ldots \]

Proceeding in this way, we will get the intrablock subgroups $B_{(p-1)p+1}$, $B_{(p-1)p+2}$, \ldots, $B_{p^2}$ placed in the last column of the construction matrix, which can be obtained as:

$B_{(p-1)p+1}$ is obtained by an addition of the vector $(-000(p-1))$ with each of the vector elements of $B_{(p-2)p+1}$;

$B_{(p-1)p+2}$ is obtained by an addition of the vector $(-000(p-1))$ with each of the vector elements of $B_{(p-2)p+1}$;

\[ \ldots \ldots \ldots \]

$B_{p^2}$ is obtained by an addition of the vector $(-000(p-1))$ with each of the vector elements of $B_{(p-1)p}$;

Thus, in the construction matrix, we get $p^2$ intrablock subgroups in $p$ columns. The method is illustrated with two examples described in the section below.

**IV. Illustrations**

Example 1. Suppose we like to construct the layout of a $3^4$ - factorial experiment where the factorial effects $ABCD$ and $ABCD^2$ are confounded simultaneously.

The plan is given by matrix, $M = [M_0 \ M_1 \ M_2]_{27\times12}$:

$M_u = [V_1(3^0) \ V_2(3^1) \ V_3(3^2) \ a_u]; u = 0, 1, 2$; with

$V_1[1] = [0101 1100 2202 2202]$ and $V_2[2] = [0110 1110 2222 2222]$

$V_3[9] = [0000 0202 0202 2222]$

The adjustment vector, $a_u = [a_u0 \ a_u1 \ \ldots \ a_u7]$ could be obtained by solving the symbolic equation corresponding to the factorial effect to be confounded.

Step 1. Find the matrices $M_0$ and $M_1$, which are the key incomplete blocks confounded with $ABCD$ and $ABCD^2$ respectively in a $3^4$ - factorial experiment.

$$
M_0 = \begin{bmatrix}
0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 2 & \ldots & 0 & 0 & 2 & 1 & 0 & 1 & 0 \\
0 & 0 & 2 & 1 & \ldots & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & \ldots & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 2 & 0 & \ldots & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 2 & 0 & 1 & \ldots & 0 & 2 & 0 & 1 & 1 & 0 & 0 \\
0 & 2 & 1 & 0 & \ldots & 0 & 2 & 1 & 0 & 0 & 1 & 1 \\
0 & 2 & 2 & 2 & \ldots & 0 & 2 & 2 & 2 & 1 & 0 & 0 \\
1 & 0 & 0 & 2 & \ldots & 1 & 0 & 0 & 2 & 0 & 0 & 1 \\
1 & 0 & 1 & 1 & \ldots & 1 & 0 & 1 & 1 & 1 & 0 & 2 \\
1 & 0 & 2 & 0 & \ldots & 1 & 0 & 2 & 0 & 2 & 0 & 1 \\
1 & 1 & 0 & 0 & \ldots & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & \ldots & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 2 & 2 & \ldots & 1 & 1 & 2 & 2 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & \ldots & 1 & 2 & 0 & 0 & 1 & 1 & 0 \\
1 & 2 & 1 & 2 & \ldots & 1 & 2 & 1 & 2 & 1 & 1 & 1 \\
1 & 2 & 2 & 1 & \ldots & 1 & 2 & 2 & 1 & 2 & 2 & 0 \\
2 & 0 & 0 & 1 & \ldots & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & \ldots & 2 & 0 & 1 & 0 & 2 & 0 & 0 \\
2 & 0 & 2 & 1 & \ldots & 2 & 0 & 2 & 1 & 0 & 2 & 0 \\
2 & 1 & 0 & 0 & \ldots & 2 & 1 & 0 & 0 & 1 & 2 & 0 \\
2 & 1 & 1 & 2 & \ldots & 2 & 1 & 1 & 2 & 1 & 2 & 0 \\
2 & 1 & 2 & 1 & \ldots & 2 & 1 & 2 & 1 & 2 & 2 & 0 \\
2 & 2 & 0 & 2 & \ldots & 2 & 2 & 0 & 2 & 1 & 2 & 0 \\
2 & 2 & 1 & 1 & \ldots & 2 & 2 & 1 & 1 & 2 & 2 & 0 \\
2 & 2 & 2 & 0 & \ldots & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\
2 & 2 & 2 & 2 & \ldots & 2 & 2 & 2 & 2 & 0 & 0 & 0 
\end{bmatrix}
$$

$$
M_0 = \begin{bmatrix}
1 & 1 & 1 & 0 & \ldots & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 2 & 2 & \ldots & 1 & 1 & 2 & 2 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & \ldots & 1 & 2 & 0 & 0 & 1 & 0 & 0 \\
1 & 2 & 1 & 2 & \ldots & 1 & 2 & 1 & 2 & 2 & 0 & 0 \\
1 & 2 & 2 & 1 & \ldots & 1 & 2 & 2 & 1 & 2 & 2 & 0 \\
2 & 0 & 0 & 1 & \ldots & 2 & 0 & 0 & 1 & 0 & 0 & 0 \\
2 & 0 & 1 & 0 & \ldots & 2 & 0 & 1 & 0 & 2 & 0 & 0 \\
2 & 0 & 2 & 1 & \ldots & 2 & 0 & 2 & 1 & 0 & 2 & 0 \\
2 & 1 & 0 & 0 & \ldots & 2 & 1 & 0 & 0 & 1 & 2 & 0 \\
2 & 1 & 1 & 2 & \ldots & 2 & 1 & 1 & 2 & 1 & 2 & 0 \\
2 & 1 & 2 & 1 & \ldots & 2 & 1 & 2 & 1 & 2 & 2 & 0 \\
2 & 2 & 0 & 2 & \ldots & 2 & 2 & 0 & 2 & 1 & 2 & 0 \\
2 & 2 & 1 & 1 & \ldots & 2 & 2 & 1 & 1 & 2 & 2 & 0 \\
2 & 2 & 2 & 0 & \ldots & 2 & 2 & 2 & 0 & 0 & 0 & 0 \\
2 & 2 & 2 & 2 & \ldots & 2 & 2 & 2 & 2 & 0 & 0 & 0 
\end{bmatrix}
$$

Step 2. Selecting the common vector elements from $M_0$ and $M_1$, we will get the key intrablock subgroup $B_1$, which is given by

$$
B_1 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 2 & 0 \\
0 & 2 & 1 & 0 \\
1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 \\
1 & 2 & 0 & 0 \\
2 & 0 & 1 & 0 \\
2 & 1 & 0 & 0 \\
2 & 2 & 2 & 0 
\end{bmatrix}
$$

Step 3. Add the vector $00110$ to each of the vector elements of the key intrablock subgroup, we get,
Thus, we have the complete layout of nine intrablock subgroups as shown below.

\[
\begin{align*}
B_2 &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 2 & 1 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 1 & 1 & 0 \\ 2 & 2 & 0 & 0 \\ \end{bmatrix} \\
B_3 &= \begin{bmatrix} 0 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 \\ 2 & 0 & 0 & 0 \\ 2 & 1 & 2 & 0 \\ 2 & 2 & 1 & 0 \\ \end{bmatrix}
\end{align*}
\]

To obtain the third intrablock subgroup \(B_3\), add the vector \((0\,0\,2\,0)\) to each of the vector elements of the key intrablock subgroup. Thus,

\[
\begin{align*}
B_4 &= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 2 & 1 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 1 & 1 & 0 \\ 2 & 2 & 0 & 0 \\ \end{bmatrix} \\
B_5 &= \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 \\ 2 & 0 & 0 & 1 \\ 2 & 1 & 2 & 1 \\ 2 & 2 & 1 & 0 \\ \end{bmatrix} \\
B_6 &= \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 2 & 1 & 1 \\ 2 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 2 & 2 & 0 & 0 \\ \end{bmatrix}
\end{align*}
\]

The intrablock subgroups \(B_4, B_5, B_6\) of second column and \(B_7, B_8, B_9\) of the third (last) column can be computed as shown below.

\(B_4\) is obtained by adding the vector \((0\,0\,0\,1)\) to each of the row vectors of \(B_1\);

\(B_5\) is obtained by adding the vector \((0\,0\,0\,1)\) to each of the row vectors of \(B_2\);

\(B_6\) is obtained by adding the vector \((0\,0\,0\,1)\) to each of the row vectors of \(B_3\); and

\(B_7\) is obtained by adding the vector \((0\,0\,0\,2)\) to each of the vectors of \(B_1\);

\(B_8\) is obtained by adding the vector \((0\,0\,0\,2)\) to each of the vectors of \(B_2\);

\(B_9\) is obtained by adding the vector \((0\,0\,0\,2)\) to each of the vectors of \(B_3\);

Thus, we have the complete layout of nine intrablock subgroups as shown below.

\[
\begin{align*}
\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 2 & 1 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 1 & 1 & 0 \\ 2 & 2 & 0 & 0 \\ \end{bmatrix} & \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 2 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ \end{bmatrix} & \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 2 & 1 & 1 \\ 1 & 0 & 2 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 1 & 1 \\ 2 & 0 & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 2 & 2 & 0 & 0 \\ \end{bmatrix} & \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 \\ 2 & 0 & 0 & 1 \\ 2 & 1 & 2 & 1 \\ 2 & 2 & 1 & 0 \\ \end{bmatrix}
\end{align*}
\]

It is easy to verify that,

\((B_1 + B_2 + B_3)Vs. (B_4 + B_5 + B_6) Vs. (B_7 + B_8 + B_9)\); confounds the 1st effect, \(ABCD\); (column comparison)

\((B_1 + B_4 + B_7)Vs. (B_2 + B_5 + B_8) Vs. (B_3 + B_6 + B_9)\); confounds the 2nd effect, \(ABC D^2\); (row comparison)

\((B_1 + B_4 + B_7)Vs. (B_2 + B_5 + B_7) Vs. (B_3 + B_6 + B_8)\); confounds the 1st generalized effect, \(ABCD + ABC D^2 = ABC\) (comparing I-totals); and

\((B_1 + B_4 + B_7) Vs. (B_2 + B_5 + B_7) Vs. (B_3 + B_6 + B_8)\); confounds the 2nd generalized effect, \(ABC D + (ABC D^2)^2 = D\) (comparing J-totals).

Example 2. Suppose we are to construct a \(5^3\) factorial experiment where the factorial effects \(ABC\) and \(ABC^2\) are confounded simultaneously.

Solution. First, we find the two incomplete blocks \(M_0\) and \(M_0'\) corresponding confounded effects \(ABC\) and \(ABC^2\).

The key incomplete block \(M_0\) confounded with \(ABC\) is given by,

\[M_0 = [V_1(5^0), V_2(5^1), a_u]_{5^{x-1} x_3}\text{, with}\]

\[V_1(5^0) = 5^015115215351451; \quad V_2(5^1) = 5^1011151531415; \quad \text{and the adjustment vector } a_u \text{ can be obtained by solving the symbolic equation } x_1 + x_2 + x_3 = 0 \mod 5, \text{ taking first two}\]
values of \( x_1 \) and \( x_2 \) from the vectors \( V_1 \) and \( V_2 \) respectively for \( ABC \).

Similarly, the key incomplete block \( M_0' \) confounded with \( ABC^2 \) is given by,
\[
M_0' = [V_1(S^3), V_2(S^3), a_u]_{k=1, \ldots, 5},
\]
and the adjustment vector \( a_u \) can be obtained by solving the symbolic equation \( x_1 + x_2 + 2x_3 = 0 \mod 5 \), taking first two values of \( x_1 \) and \( x_2 \) from the vectors \( V_1 \) and \( V_2 \) respectively for \( ABC^2 \).

\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 4 \\
0 & 2 & 3 \\
0 & 3 & 2 \\
0 & 4 & 1 \\
1 & 0 & 4 \\
1 & 1 & 3 \\
1 & 2 & 2 \\
1 & 3 & 1 \\
1 & 4 & 0 \\
2 & 0 & 3 \\
2 & 1 & 2 \\
2 & 2 & 1 \\
2 & 3 & 0 \\
2 & 4 & 4 \\
3 & 0 & 2 \\
3 & 1 & 1 \\
3 & 2 & 0 \\
3 & 3 & 4 \\
3 & 4 & 3 \\
4 & 0 & 1 \\
4 & 1 & 0 \\
4 & 2 & 4 \\
4 & 3 & 3 \\
4 & 4 & 2
\end{bmatrix}

\[
M_0 = \begin{bmatrix}
0 & 0 & 0 \\
0 & 1 & 2 \\
0 & 2 & 4 \\
0 & 3 & 1 \\
0 & 4 & 3 \\
1 & 0 & 2 \\
1 & 1 & 4 \\
1 & 2 & 1 \\
1 & 3 & 3 \\
1 & 4 & 0 \\
2 & 0 & 4 \\
2 & 1 & 1 \\
2 & 2 & 3 \\
2 & 3 & 0 \\
2 & 4 & 2 \\
3 & 0 & 1 \\
3 & 1 & 3 \\
3 & 2 & 0 \\
3 & 3 & 2 \\
3 & 4 & 4 \\
4 & 0 & 3 \\
4 & 1 & 0 \\
4 & 2 & 2 \\
4 & 3 & 4 \\
4 & 4 & 1
\end{bmatrix}

Step 1.

Select the common elements (vectors) of \( M_0 \) and \( M_0' \) to find the key intrablock subgroup \( B_1 \), as shown below.

\[
B_1 = \begin{bmatrix}
0 & 0 & 0 \\
1 & 4 & 0 \\
2 & 3 & 0 \\
3 & 2 & 0 \\
4 & 1 & 0
\end{bmatrix}
\]

Step 2.

We get the second intrablock subgroup \( B_2 \) by an addition of the vector \( (0 \ 0 \ 1) \) to each of the vectors of the key intrablock subgroup; \( B_2 \) is placed just below (column side) the key block \( B_1 \).

Similarly, we get the third, fourth and fifth intrablocks \( B_3, B_4 \) and \( B_5 \) by addition the vectors \((0 \ 2 \ 0), (0 \ 3 \ 0)\) and \((0 \ 4 \ 0)\) to each of the vectors of the key intrablock subgroups respectively, shown below.

Step 3.

After getting the intrablock subgroups, \( B_1, B_2, B_3, B_4 \) and \( B_5 \) we get the intrablock subgroups \( B_6, B_7, B_8, B_9, B_10 \) by an addition of the vector \((0 \ 0 \ 1)\) to each of the vector elements (row vectors) of \( B_1, B_2, B_3, B_4 \) and \( B_5 \) respectively.

An addition of the vector \((0 \ 0 \ 2)\) to each of the vector elements of \( B_1, B_2, B_3, B_4, B_6, B_7, B_8, B_9, B_10, B_11, B_12, B_13, B_14, B_15 \) will produce the intrablock subgroups \( B_6, B_7, B_8, B_9, B_10 \) and \( B_20 \); and

An addition of the vector \((0 \ 0 \ 3)\) to each of the vector elements of \( B_1, B_2, B_3, B_4, B_5, B_6, B_7, B_8, B_9, B_10, B_11, B_12, B_13, B_14, B_15, B_16, B_17, B_18, B_19, B_20 \) will produce the intrablock subgroups \( B_21, B_22, B_23, B_24, B_25 \), which completes the plan, shown below.
V. Conclusion
In this article, a general construction method has been developed for simultaneous confounding in $p^n$ ($p$ is prime) factorial experiment. The construction of simultaneous confounding in $p^n$ factorial experiment becomes easier and rewarding. The method is restricted to $p^n$ factorial experiment when $p$ is prime and it can be extended for a simultaneous confounding of three or more factorial effects.

References: