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# The General Solution of Minisuperspace Wheeler-De Witt Equation

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## Abstract

Wheeler-De Witt (WDW) equation is the central equation of canonical quantum gravity. For its infinite dimensionality to get a meaningful solution of that equation is very hard. To extract minimal information, one has to consider WDW equation in Minisuperspace where there are only two variables, such as known radius of the Universe and the scalar field. In this paper we find the general solution of WDW equation in Minisuperspace. We discuss the physical significance of our solution.

*Keywords: Quantum gravity, Wheeler De Witt equation, Minisuperspace, Exact solution*

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হুইলার-ডি উইট সমীকরণ ক্যানোনিক্যাল কোয়ান্টাম গ্রাভিটির কেন্দ্রীয় সমীকরণ। এটির অসীম সংখ্যক মাত্রা থাকার কারণে এ সমীকরণ থেকে কোন অর্থপূর্ণ সমাধান পাওয়া খুব কঠিন। ন্যূনতম তথ্য পেতে হুইলার-ডি উইট সমীকরণকে দ্বিমাত্রিক মিনিসুপারস্পেসে বিবেচনা করা হয় যেখানে দু'টি মাত্রা হলো মহাবিশ্বের জানা ব্যাসার্ধ এবং স্কেলার ক্ষেত্র। এ প্রবন্ধে আমরা মিনিসুপারস্পেসে হুইলার-ডি উইট সমীকরণের সাধারণ সমাধান বের করেছি। আমাদের সমাধানের ভৌত তাৎপর্য আলোচনা করেছি।

## 1. Introduction

It's a longstanding problem to unify Quantum Mechanics and General Theory of Relativity. The first one deals with the very small structure of the matter, whereas the later deals with the very large structure of the Universe. It seems these two theories are incompatible indeed. But a number of scientists do feel that these two theories must be united in the viewpoint of theoretical beauty. It is thought that in the very beginning of the Universe it has undergone an exponentially expansion from a singular point, popularly known as *Big Bang* model [1-4]. The primordial ultra-super dense particle had almost infinite energy within almost zero volume. In this case the quantum fluctuation might be so violent that at the beginning of the Universe we can consider the application of quantum mechanics.

Let us consider the quantum state of a spatially closed Universe can be described by a wave function $\Psi[h_{ij}]$ which is a functional of three geometries $h_{ij}$. This wave function satisfies a functional differential Schrödinger-like equation which is known as WDW equation [5,6]

$$\left\{-G_{ijkl}\frac{\delta^2}{\delta h_{ij}\,\delta h_{kl}} - \sqrt{h}\;{}^3R\right\}\Psi\left[h_{ij}\right] = 0 \tag{1}$$

where ${}^3R$ is three-dimensional Ricci scalar, $G_{ijkl}$ is known as De Witt metric of superspace or supermetric and $G_{ijkl}$ is

$$G_{ijkl} = \frac{1}{2\sqrt{h}}\left(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl}\right) \tag{2}$$

The inverse De Witt metric is given by

$$G^{ijkl} = \frac{\sqrt{h}}{2}\left(h^{ik}h^{jl} + h^{il}h^{jk} - h^{ij}h^{kl}\right) \tag{3}$$

It is clear from equation (2) and (3) that

$$G^{ijkl}G_{ijkl} = \frac{1}{2}\left(\delta^i_m\delta^j_n + \delta^i_n\delta^j_m\right) \tag{4}$$

## 2. Superspace and Minisuperspace

The three-metric $h_{ij}$ is the functions of $x^1, x^2, x^3$and it determines the distance $d\sigma$ between infinitesimally separated points $(x^1, x^2, x^3)$ and $(x^1 + \Delta x^1, x^2 + \Delta x^2, x^3 + \Delta x^3)$ as follows:

$$d\sigma^2 = h_{ij}\Delta x^i \Delta x^j \tag{5}$$

This distance is invariant under spatial coordinate transformations. In the similar manner, the space of all metrices $h_{ij}$ can be regarded as a *superspace* in which the points are the metric functions of $h_{ij}$. Now one can define a corresponding supermetric $G_{ijkl}$ so that the infinitesimal distance $d\Sigma$ between neighboring metric $h_{ij}$ and $(h_{ij}+\Delta h_{ij})$ is given by

$$d\Sigma^2 = G_{ijkl}\delta h^{ij}\delta h^{kl} \tag{6}$$

which is invariant in a suitable sense when transformation $h^{ij} \rightarrow h'^{ij}$ is considered. Due to the difficulty of the superspace for its infinite dimensionality, we simplify the problem considering a *Minisuperspace* which is defined by homogeneous and isotropic manifolds where Euclidean histories can contribute to the sum defining the wavefunction [5-7]. The suitable metric of this form can be given by

$$ds^2 = \sigma^2[N^2(t)dt^2 + a^2(t)\, d\Omega_3^2] \tag{7}$$

where $N(t)$ is the lapse function, $\sigma^2 = \frac{l^2}{24\pi^2}$, $l^2 = 16\pi = 2\kappa^2$ and $d\Omega_3$ is the line element on the three-sphere $S_3$, given by

$$d\Omega_3^2 = d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{8}$$

The Euclidean action can be written as [7]

$$l^2 I_E = -\int_V d^4x\, g^{1/2}\,({}^4R - 2\Lambda) + 2\int_{\partial V} d^3x\, h^{1/2}K^2 \tag{9}$$

The first term is integrated over spacetime and the second term over its boundary. Here, $K$ is the trace of the intrinsic curvature of the boundary three surface [7-9]. Expressing the four-dimensional curvature ${}^4R$ into the sum of three-dimensional spatial curvature ${}^3R$ and intrinsic curvature tensor $K_{ij}$ and neglecting the surface terms of action, we get

$$l^2 I_E = -\int_V d^4x\, g^{\frac{1}{2}}\left[{}^3R + K^2 - K_{ij}K^{ij} - 2\Lambda\right] \tag{10}$$

From equation (7) we calculate

$${}^3R = \frac{6}{\sigma^2 a^2}\,; \qquad K^2 - K_{ij}K^{ij} = \frac{6}{\sigma^2}\left(\frac{1}{aN}\right)^2\left(\frac{da}{d\tau}\right)^2 \tag{11}$$

where $\tau$ is the proper time. The volume of the three-sphere [7] can be given by

$$\int d^3x\, h^{1/2} = 2\pi^2(\sigma a)^3 \tag{12}$$

Hence,

$$I_E = \frac{1}{2}\int d\tau \ \left(\frac{N}{a}\right)\left[-\left(\frac{a\dot{a}}{N}\right)^2 - a^2 + \lambda a^4\right] \tag{13}$$

where $\frac{\sigma^2\Lambda}{3} = H^2 = \lambda,\quad \dot{a} = da/d\tau$, $H$ is the Hubble constant and $\lambda$ is a parameter.

Let us now include a conformally invariant scalar field $\phi$ to represent the matter degrees of freedom. The conformal scalar field has the action

$$I_\varphi = \frac{1}{2}\int_V \ d^4x \left[(\nabla\varphi)^2 + \ {}^4R\ \varphi^2/6\right] + \frac{1}{12}\int_{\partial V} d^3x\ h^{1/2}K \tag{14}$$

The boundary term is cancelled by ${}^4R\ \varphi^2/6$. The classical field equation for $\varphi$ is

$$g^{\mu\upsilon}\varphi_{;\mu\upsilon} - \frac{{}^4R\varphi}{6} = 0 \tag{15}$$

which is conformally invariant under transformation

$$g^{\mu\upsilon}(x) \to \Omega^2(x) g^{\mu\upsilon}(x) \tag{16}$$

$$\varphi(x) \to \Omega^{-1}(x)\ \ \varphi(x) \tag{17}$$

After rescaling

$$\varphi = \frac{\chi}{(2\pi^2\sigma^2)^{1/2}a} \tag{18}$$

One obtains Lorentzian action keeping $\chi$ and $a$ fixed on the boundary

$$I = I_E + I_\varphi = \frac{1}{2}\int dt\ \left[\frac{N}{a}\right]\left[\left[\frac{a\,\dot{a}}{N}\right]^2 + a^2 - \lambda a^4 + \left[\frac{a}{N}\dot{\chi}\right] - \chi^2\right] \tag{19}$$

where$\chi$ is conformally invariant. Following standard procedure, the momenta $\pi_a$ and $\pi_\chi$ can be constructed as follows:

$$\frac{1}{2}\left(-\pi_a^2 - a^2 + \lambda a^4 + \pi_\chi^2 + \chi^2\right) = 0 \tag{20}$$

This equation can be expressed in operator form. For matter-energy renormalization an arbitrary constant, say, $2\epsilon_0$ can be included into this equation. There the WDW equation can be given by

$$\frac{1}{2}\left[\frac{1}{a^p}\frac{\partial}{\partial a}\left[a^p\frac{\partial}{\partial a}\right] - (a^2 - \chi^2)\Psi + \lambda a^4 - \frac{\partial^2}{\partial\chi^2} - 2\epsilon_0\right]\Psi(a,\chi) = 0 \tag{21}$$

## 3. Exact Solution

To simplify the problem let us put, $\lambda = 0$ and $p = 1$ in equation (21), we have

$$\frac{1}{a}\frac{\partial\Psi}{\partial a} + \frac{\partial^2\Psi}{\partial a^2} - (a^2 - \chi^2)\Psi - \frac{\partial^2\Psi}{\partial\chi^2} - 2\epsilon_0\Psi = 0 \qquad (22)$$

Let $\zeta = (a^2 - \chi^2)$and $\Psi = \Psi(\zeta)$. Then

$$\frac{\partial\Psi}{\partial a} = \frac{\partial\Psi}{\partial\zeta}\frac{\partial\zeta}{\partial a} = 2a\Psi' \qquad (23)$$

where $\Psi' = \partial\Psi/\partial\zeta$.

Let us calculate the followings

$$\frac{\partial^2\Psi}{\partial a^2} = \frac{\partial}{\partial a}(2a\Psi') = 2\Psi' + 4a^2\Psi'' \qquad (24a)$$

$$\frac{\partial\Psi}{\partial\chi} = \frac{\partial\Psi}{\partial\zeta}\frac{\partial\zeta}{\partial\chi} = -2\chi\Psi' \qquad (24b)$$

$$\frac{\partial^2\Psi}{\partial\chi^2} = \frac{\partial}{\partial\chi}\frac{\partial\Psi}{\partial\chi} = -2\Psi' + 4\chi^2 \qquad (24c)$$

Putting all these values in equation (22), we get after simplification

$$4\zeta\Psi'' + 6\Psi' - \zeta\Psi - 2\epsilon_0\Psi = 0 \qquad (25)$$

Let us consider the following transformation

$$\Psi = Ae^{\alpha\zeta}\, u(\zeta) \qquad (26)$$

where $A$ and $\alpha$ is are arbitrary constants.

Then we calculate the followings

$$\Psi' = A\alpha e^{\alpha\zeta}u + Ae^{\alpha\zeta}u' \qquad (27a)$$

$$\Psi'' = A\alpha^2 e^{\alpha\zeta}u + 2A\alpha e^{\alpha\zeta}u' + Ae^{\alpha\zeta}u'' \qquad (27b)$$

Putting all these in equation (25), we get

$$4\zeta\left[A\alpha^2 e^{\alpha\zeta}u + 2A\alpha e^{\alpha\zeta}u' + Ae^{\alpha\zeta}u''\right] + 6\left[A\alpha e^{\alpha\zeta}u + Ae^{\alpha\zeta}u'\right] - \zeta Ae^{\alpha\zeta}\, u - 2\epsilon_0 Ae^{\alpha\zeta}\, u = 0 \qquad (28)$$

To simplify the problem let us take $4\alpha^2 = 1$and $6\alpha = 2\epsilon_0$. Then dividing by $2Ae^{\alpha\zeta}$ equation (28) becomes

$$2\zeta u'' + 4\alpha\zeta u' + 3u' = 0 \qquad (29)$$

Rearranging, we have

$$\frac{u''}{u'} = -\frac{4\zeta u+3}{2\zeta} \tag{30}$$

By integrating the equation (30), we obtain

$$\ln u' = -2\alpha\zeta - \frac{3}{2} ln\, \zeta + const. \tag{31}$$

$$u' = \frac{du}{d\zeta} = B\, e^{-2\alpha\zeta}\zeta^{-3/2} \tag{32}$$

where $B$ is a constant. Therefore

$$u = B \int_{\zeta_0}^{\zeta} e^{-2\alpha\zeta}\zeta^{-3/2}\, d\zeta \tag{33}$$

Now we get the solution for $\Psi$ is

$$\Psi = Ae^{\alpha\zeta}u = C\, e^{\alpha\zeta}\int_{\zeta_0}^{\zeta} e^{-2\alpha\zeta}\zeta^{-3/2}\, d\zeta \tag{34}$$

where $C$ is a constant. This provides the general solution of the Minisuperspace Wheeler-De Witt equation. The probability of the universe being in a small interval $(a+\partial a)$ and $(\chi+\partial\chi)$ is given by

$$P \propto \int |\Psi|^2 \delta a\, \delta\chi \tag{35}$$

The relative probability can be defined by

$$\mathcal{P} = \frac{a_1|\Psi(a_1,\chi_1)|^2}{a_2|\Psi(a_2,\chi_2)|^2} = \frac{a_1}{a_2} \tag{36}$$

on $\chi =$constant. $\chi$ is related to scalar field $\varphi$ by $\chi = ka\varphi$, where $k$ is a constant. Therefore,

$$\zeta = a^2 - \chi^2 = a^2(1-k^2\varphi^2) \tag{37}$$

The coordinates $(a, \varphi)$ are depicted in the following figure 1:

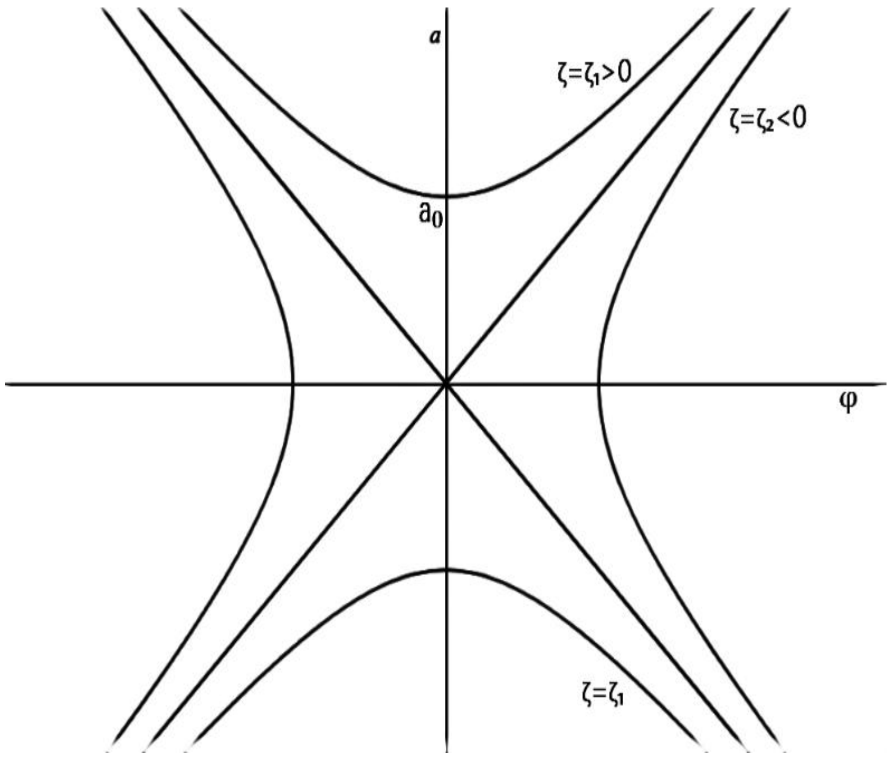


**Figure 1.** Coordinates $(a, \varphi)$ depicted the equation (37)

## 4. Discussion

Equation (34) provides the general solution of Minisuperspace WDW equation. The integral in equation (34) gives a constant for suitable values. So, we can fairly demand that at the present state the Universe is expanding exponentially for $\alpha = 1/2$. The integral diverges when $\mathrm{a} = 0$ . It signifies that the Universe started from a non-zero radius. One can assume that non-zero radius might be the Planck length. When $\alpha = -1/2$, the Universe contracts exponentially. The phase change between expanding and contracting Universe cannot be described by our model. In a paper [3] we also provided an exact solution of WDW equation which also supports the exponentially expanding and contracting Universe. For better understanding one might consider these papers [10, 11]. Overall, our model suggests an exponentially oscillating Universe in the simplest form.

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