

On the Equation $Z(n) + SL(n) = n$

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Abstract

This paper considers the Diophantine equation $Z(n) + SL(n) = n$, where $Z(n)$ is the pseudo Smarandache function and $SL(n)$ is the Smarandache LCM function.

Keywords: Pseudo Smarandache function, Smarandache LCM function, Diophantine equation

GB wbe†Ü Diophantine mgxKiY $Z(n)+SL(n)=n$ we†ePbv Kiv ntq†Q, thLv†b $Z(n)$ n†j v Pseudo †§ivÜv†P dvskb Ges $SL(n)$ n†j v †§ivÜv†P Gj .wm.Gg. A†cyK|

1. Introduction

The pseudo Smarandache function $Z(n)$, introduced by Kashihara [1], is defined as follows :

$$Z(n) = \min \left\{ m : n \mid \frac{m(m+1)}{2} \right\}.$$

The Smarandache LCM function, denoted by $SL(n)$, is defined as

$$SL(n) = \min \{ k \geq 1 : n \mid [1, 2, \dots, k] \},$$

where $[1, 2, \dots, k]$ is the least common multiple of the integers $1, 2, \dots, k$.

Then, we have the following result.

Lemma 1.1. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ be the (unique) representation of the integer n in terms of its r prime factors p_1, p_2, \dots, p_r . Then,

$$SL(n) = \max \{ p_1^{\alpha_1}, p_2^{\alpha_2}, \dots, p_r^{\alpha_r} \}.$$

In a paper, Xin Xu [2] considers the Diophantine equation

$$Z(n) + SL(n) = n.$$

In this paper, we follow a simple approach to the solution of the above equation.

2. Main Result

Though a closed-form expression of $SL(n)$ is available, given in Lemma 1.1, the general form of $Z(n)$ has not yet been found. The expressions of $Z(n)$ for some particular cases of n , obtained so far, are given in Majumdar [3-5]. For a brief review of the pseudo Smarandache function, we refer the reader to Liu [6]. Theorem 4.2.2 in [3] gives a method of finding $Z(pq)$, where p and $q (> p)$ are distinct primes. We follow the same method to find $Z(2^k p^\alpha)$ for some special cases. This is done in the lemma below.

Lemma 2.1. Let the integer n be of the form

$$n = 2^k p^\alpha,$$

where $p \geq 3$ is a prime, and $k \geq 1$ and $\alpha \geq 1$ are integers. Then,

- (1) if $2^{k+1} \mid (p^\alpha - 1)$, then $Z(n) = p^\alpha - 1$,
- (2) if $2^k \mid (p^\alpha - 1)$ but 2^{k+1} does not divide $(p^\alpha - 1)$, then $Z(n) = p^\alpha(2^k - 1)$,
- (3) if $p^\alpha \mid (2^k - 1)$, then $Z(n) = 2^k(p^\alpha - 1)$.

Proof. Since

$$Z(n) = Z(2^k p^\alpha) = \min \left\{ m : 2^k p^\alpha \mid \frac{m(m+1)}{2} \right\},$$

there are two possibilities :

Case 1. $2^{k+1} \mid m, p^\alpha \mid (m+1)$.

In this case,

$$m = 2^{k+1} x \text{ for some integer } x \geq 1,$$

$$m + 1 = p^\alpha y \text{ for some integer } y \geq 1.$$

This leads to the following Diophantine equation :

$$p^\alpha y - 2^{k+1} x = 1. \tag{1}$$

Case 2. $2^{k+1} \mid (m+1), p^\alpha \mid m$.

Then,

$$m + 1 = 2^{k+1} x \text{ for some integer } x \geq 1,$$

$$m = p^\alpha y \text{ for some integer } y \geq 1,$$

so that the resulting Diophantine equation is

$$2^{k+1} x - p^\alpha y = 1. \tag{2}$$

(1) Let $2^{k+1} \mid (p^\alpha - 1)$. Then,

$$p^\alpha - 1 = 2^{k+1} a \text{ for some integer } a \geq 1.$$

Then, the Diophantine equations (1) and (2) take the forms

$$(2^{k+1} a + 1)y - 2^{k+1} x = 1,$$

$$2^{k+1} x - (2^{k+1} a + 1)y = 1,$$

that is,

$$2^{k+1}(ay - x) + y = 1, \quad (3)$$

$$2^{k+1}(x - ay) - y = 1. \quad (4)$$

Therefore, the minimum solution is obtained from (3) with

$$y = 1, \quad ay - x = 0.$$

Hence, the minimum m is given by

$$m = p^\alpha - 1.$$

(2) Let 2^k divide $(p^\alpha - 1)$ but 2^{k+1} does not divide $(p^\alpha - 1)$. Then,

$$p^\alpha - 1 = 2^k b \text{ for some integer } b \geq 1, \quad b \neq 2.$$

Therefore, the Diophantine equations (1) and (2) can be written as

$$(2^k b + 1)y - 2^{k+1}x = 1,$$

$$2^{k+1}x - (2^k b + 1)y = 1,$$

that is,

$$2^k(by - 2x) + y = 1, \quad (5)$$

$$2^k(2x - by) - y = 1. \quad (6)$$

In this case, the minimum solution is obtained from (6) as follows:

$$2x - by = 1, \quad y = 2^k - 1,$$

Thus, the minimum m is

$$m = p^\alpha y = p^\alpha (2^k - 1).$$

(3) Let $p^\alpha / (2^k - 1)$.

Here,

$$2^k - 1 = p^\alpha c \text{ for some integer } c \geq 1.$$

Then, the Diophantine equations (1) and (2) become

$$p^\alpha y - 2(p^\alpha c + 1)x = 1,$$

$$2(p^\alpha c + 1)x - p^\alpha y = 1,$$

that is,

$$p^\alpha (y - 2cx) - 2x = 1, \tag{7}$$

$$p^\alpha (2cx - y) + 2x = 1. \tag{8}$$

The minimum solution, obtained from (7) is

$$y - 2cx = 1, 2x = p^\alpha - 1.$$

Consequently, the minimum m is

$$m = 2^{k+1} x = 2^k (p^\alpha - 1).$$

All these complete the proof of the lemma.

We now have the following theorem, giving the solution of the Diophantine equation

$$Z(n) + SL(n) = n.$$

Theorem 2.1. The Diophantine equation $Z(n) + SL(n) = n$ has the solution

$$n = 2^k p^\alpha,$$

where 2^k and p^α satisfy one of the two conditions (2) and (3) of Lemma 2.1

Proof. is evident from Lemma 2.1.

3. Conclusions

In the paper, we have obtained the solution of the Diophantine equation

$$Z(n) + SL(n) = n \text{ in the form } n = 2^k p^\alpha$$

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