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On the Closed Forms of $Z(pq)$, $q = kp \pm 10$

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Abstract

This paper derives the closed form expressions of $Z(pq)$, where p and q ($> p$) are primes, and q is of the forms $q = kp \pm 10$ for some (positive) integer k, and $Z(.)$ is the pseudo Smarandache function.

Keywords: *Pseudo Smarandache function, Diophantine equation.*

GB wbe‡Ü $Z(pq)$ Gi Ave× i/c iwk¸‡jv wb®úbœKiv n‡q‡Q, †hLv‡b p Ges q $(>P)$ †gŠwjK nq Ges k n‡jv wKQz(abvZ¥K) c~Y©msL¨v, Ges q Gi Rb¨ $q=kp\pm 10$ i/c Ges $Z(.)$ nj Pseudo ¯§vivÛv‡P A‡cÿK|

1. Introduction

The pseudo Smarandache function $Z(n)$, introduced by Kashihara [1], is follows :

$$Z(n) = min\{m : n \mid \frac{m(m+1)}{2}\}.$$

The expressions of $Z(n)$ for some particular cases of n are given by Kashihara [1], Asbacher [2] and Majumdar [3 – 5]. A method of finding $Z(pq)$ is given in Theorem 4.2.2 in [3] (where p and q (> p) are primes) which is outlined below : Since

$$Z(p\,q) = min\{m : p\,q \mid \frac{m(m+1)}{2}\},$$

there are two possibilities :

Case 1. $p \mid m,\ q \mid (m+1)$.

In this case,

$$m = p\,x \text{ for some integer } x \geq 1,$$

$$m + 1 = q\,y \text{ for some integer } y \geq 1.$$

This leads to the following Diophantine equation:

$$qy - px = 1. \tag{1.1}$$

Case 2. $p \mid (m+1),\ q \mid m$.

Here,

$$m + 1 = p\,x \text{ for some integer } x \geq 1,$$

$$m = q\,y \text{ for some integer } y \geq 1,$$

so that the resulting Diophantine equation is

$$px - qy = 1. \tag{1.2}$$

Thus, the problem of finding $Z(pq)$ reduces to the problem of solving the Diophantine equations (1.1) and (1.2) for minimum values of x or y, as the case may be.

In [3], Majumdar gives the expressions of $Z(pq)$, where p and q (> p) are primes and q is of the forms $q = kp + \ell$ and $q = (k+1)p - \ell$, $1 \leq \ell \leq 8$. In this paper, we give an expression of $Z(pq)$, when q is of the form $q = kp + 10$, $k \geq 2$ or $q = (k+1)p - 10$,

$k \geq 2$, to supplement the results in [3]. This is done in Section 2 and Section 3 respectively.

2. Values of $Z(pq)$, $q = kp + 10$

In this section, we find an explicit form of $Z(pq)$, where p and q are primes with $q = kp + 10$ for some integer $k \geq 2$.

First note that, when $q = kp + 10$, $k \geq 2$, the Diophantine equations (1.1) and (1.2) become

$$(kp + 10)y - px = 1,$$

$$px - (kp + 10)y = 1,$$

that is,

$$10y - (x - ky)p = 1, \quad (2.1)$$

$$(x - ky)p - 10y = 1. \quad (2.2)$$

The lemma below gives the closed-form expression of $Z(pq)$, where $q = kp + 10$.

Lemma 2.1. Let p and q ($> p$) be two primes; moreover, let q be of the form $q = kp + 10$ for some integer $k \geq 2$. Then,

$$Z(pq) = \begin{cases} \dfrac{q(p-1)}{10}, & \text{if } 10|(p-1) \\ \dfrac{q(3p+1)}{10} - 1, & \text{if } 10|(p-3) \\ \dfrac{q(3p-1)}{10}, & \text{if } 10|(p-7) \\ \dfrac{q(p+1)}{10} - 1, & \text{if } 10|(p-9) \end{cases}$$

Proof. We consider the following four cases that may arise.

Case 1. $p = 10a + 1$ for some integer $a \geq 1$.

In this case, the Diophantine equations (2.1) and (2.2) read as

$$1 = 10y - (x - ky)(10a + 1) = 10[y - (x - ky)a] - (x - ky),$$

$$1 = (x - ky)(10a + 1) - 10y = (x - ky) - 10[y - (x - ky)a].$$

Clearly, the minimum solution is obtained from the second of the above two equations with

$$x - ky = 1, \; y - (x - ky)a = 0.$$

Then, the minimum solution is $y = a$, and hence, the minimum m is given by

$$m = qy = \frac{q(p - 1)}{10}.$$

Case 2. $p = 10a + 3$ for some integer $a \geq 0$.

Here, (2.1) and (2.2) become

$$1 = 10y - (x - ky)(10a + 3) = 10[y - (x - ky)a] - 3(x - ky),$$

$$1 = (x - ky)(10a + 3) - 10y = 3(x - ky) - 10[y - (x - ky)a].$$

The minimum solution is obtained from the first of the above two Diophantine equations with

$$x - ky = 3, \;\; y - (x - ky)a = 1.$$

Therefore, the minimum y is $y = 3a + 1$, and consequently, the minimum m is

$$m = qy - 1 = q(3a + 1) - 1 = \frac{q(3p + 1)}{10} - 1.$$

Case 3. $p = 10a + 7$ for some integer $a \geq 0$.

Here, the Diophantine equations satisfied are

$$1 = 10y - (x - ky)(10a + 7) = 10[y - (x - ky)a] - 7(x - ky),$$

$$1 = (x - ky)(10a + 7) - 10y = 7(x - ky) - 10[y - (x - ky)a],$$

For which the minimum solution is obtained from the second equation as

$$x - ky = 3, \;\; y - (x - ky)a = 2.$$

Thus, the minimum solution is $y = 3a + 2$, and the minimum m is

$$m = qy = q(3a + 2) = \frac{q(3p - 1)}{10}.$$

Case 4. $p = 10a + 9$ for some integer $a \geq 1$.

In this case, the Diophantine equations (2.1) and (2.2) take the forms

$$1 = 10y - (x - ky)(10a + 9) = 10[y - (x - ky)a] - 9(x - ky),$$

$$1 = (x - ky)(10a + 9) - 10y = 9(x - ky) - 10[y - (x - ky)a].$$

Clearly, the minimum solution is obtained from the first equation, with

$$x - ky = 1, \;\; y - (x - ky)a = 1.$$

Thus, $y = a + 1$, and the minimum m is given by

$$m = qy - 1 = q(a + 1) - 1 = \frac{q(p + 1)}{10} - 1.$$

All these complete the proof of the lemma.

We now give some examples of the application of Lemma 2.1. In Case 1 in the proof of the lemma, letting $a = 1$, we get the prime $p = 11$, so that the prime q is of the form $q = 11k + 10, \;\; k \geq 1$. The first few functions in this case are

$$Z(11 \times 43) = 43, \;\; Z(11 \times 109) = 109, Z(11 \times 131) = 131.$$

The next prime is $p = 31$, which gives, for example, $Z(31 \times 41) = 123$ and $Z(31 \times 103) = 309$.

In Case 2, $a = 0$ gives the prime $p = 3$, and so, $q = 3k + 10$, $k \geq 1$. Lemma 2.1 gives

$$Z(3q) = q - 1,$$

which is true by Lemma 4.2.15 of Majumdar [3]. Thus, Case 2 holds true for $a = 0$ as well. The next prime is $p = 13$, and we get, for example,

$$Z(13\times 23) = 91, \quad Z(13\times 101) = 403, Z(13\times 127) = 507.$$

In Case 3, if $p = 7$, then $q = 7k + 10$, $k \geq 1$, and in this case, Lemma 2.1 gives

$$Z(7q) = 2q,$$

which is validated by Lemma 4.2.19 of Majumdar [3]. Thus, Case 3 holds for $a = 0$ as well. The next prime in the sequence is $p = 17$, so that $q = 17k + 10$, $k \geq 1$. The first few functions are

$$Z(17\times 61) = 305, \quad Z(17\times 163) = 815, \quad Z(17\times 197) = 985.$$

In Case 4, the first prime is $p = 19$, so that $q = 19k + 10$, $k \geq 1$. Corresponding to this case, we get the following functions:

$$Z(19\times 29) = 57, \quad Z(19\times 67) = 133, \quad Z(19\times 181) = 361.$$

The next prime is $p = 29$, so that $q = 29k + 10$, $k \geq 1$. Some of the functions in this case are

$$Z(29\times 97) = 290, \quad Z(29\times 271) = 812, \quad Z(29\times 503) = 1508.$$

3. Values of $Z(pq)$, $q = (k + 1)p – 10$

This section derives an expression of $Z(pq)$, where p and q are primes, and q is of the form $q = (k + 1)p – 10$ for some integer $k \geq 2$.

When $q = (k + 1) p – 10$, the Diophantine equations (1.1) and (1.2) become

$$[(k + 1) p – 10]y – px = 1,$$

$$px – [(k + 1) p – 10]y = 1,$$

that is,

$$1 = [(k+1)y - x]p - 10y, \tag{3.1}$$

$$1 = 10y - [(k+1)y - x]p. \tag{3.2}$$

We now prove the following lemma, giving an expression of $Z(pq)$ with $q = (k + 1)p - 10$.

Lemma 3.1. Let p and q ($> p$) be two primes with $q = (k + 1)p - 10$ for some integer $k \geq 2$. Then,

$$Z(pq) = \begin{cases} \frac{q(p-1)}{10} - 1, & \text{if } 10|(p-1) \\ \frac{q(3p+1)}{10}, & \text{if } 10|(p-3) \\ \frac{q(3p-1)}{10} - 1, & \text{if } 10|(p-7) \\ \frac{q(p+1)}{10}, & \text{if } 10|(p-9) \end{cases}$$

Proof. We consider below separately the four cases that may arise:

Case 1. $p = 10a + 1$ for some integer $a \geq 1$.

In this case, the Diophantine equations (3.1) and (3.2) may be rewritten as

$$1 = [(k+1)y - x](10a + 1) - 10y = [(k+1)y - x] - 10[y - \{(k+1)y - x\}a],$$

$$1 = 10y - [(k+1)y - x](10a + 1) = 10[y - \{(k+1)y - x\}a] - [(k+1)y - x].$$

The first of these two give the minimum solution, namely,

$$(k+1)y - x = 1,\ y - \{(k+1)y - x\}a = 0.$$

The minimum solution is thus $y = a$, and consequently, the minimum m is

$$m = qy - 1 = \frac{q(p-1)}{10} - 1.$$

Case 2. $p = 10a + 3$ for some integer $a \geq 0$.

Here, (3.1) and (3.2) become

$$1 = [(k+1)y - x](10a+3) - 10y = 3\,[(k+1)y - x] - 10[y - \{(k+1)y - x\}a],$$

$$1 = 10y - [(k+1)y - x](10a+3) = 10[y - \{(k+1)y - x\}a] - 3[(k+1)y - x].$$

The minimum solution, obtained from the second of the above two equations, is

$$(k+1)y - x = 3, \quad y - \{(k+1)y - x\}a = 1.$$

Then, the minimum solution is $y = 3a + 1$, and the minimum m is

$$m = qy = \frac{q(3p+1)}{10}.$$

Case 3. $p = 10a + 7$ for some integer $a \geq 0$.

In this case, from (3.1) and (3.2), we have

$$1 = [(k+1)y - x](10a+7) - 10y = 7\,[(k+1)y - x] - 10[y - \{(k+1)y - x\}a],$$

$$1 = 10y - [(k+1)y - x](10a+7) = 10[y - \{(k+1)y - x\}a] - 7[(k+1)y - x].$$

The minimum solution is then obtained from the first equation as follows:

$$(k+1)y - x = 3, \quad y - \{(k+1)y - x\}a = 2.$$

Thus, $y = 3a + 2$, and the minimum m is

$$m = qy - 1 = \frac{q(3p-1)}{10} - 1.$$

Case 4. $p = 10a + 9$ for some integer $a \geq 1$.

From the Diophantine equations (3.1) and (3.2), we have

$$1 = [(k+1)y - x](10a+9) - 10y = 9\,[(k+1)y - x] - 10[y - \{(k+1)y - x\}a],$$

$$1 = 10y - [(k+1)y - x](10a+9) = 10[y - \{(k+1)y - x\}a] - 9[(k+1)y - x].$$

Clearly, the minimum solution is obtained from the second equation, which is

$$(k+1)y - x = 1, \quad y - \{(k+1)y - x\}a = 1.$$

This gives the minimum solution $y = a + 1$, and the minimum m is

$$m = qy = \frac{q(p+1)}{10}.$$

In Case 1, the first prime is $p = 11$ with $q = 11(k + 1) - 10$, $k \geq 1$. Some of the functions are

$$Z(11\times 23) = 22,\ \ Z(11\times 67) = 66,\ \ Z(11\times 89) = 88.$$

The next prime of the sequence is $p = 31$, so that $q = 31(k + 1) - 10$, $k \geq 1$. The first such q is $q = 83$, with $Z(31\times 83) = 248$. The next function is $Z(31\times 269) = 806$.

In Case 2, the first prime is $p = 3$ with $q = 3(k + 1) - 10$, $k \geq 1$. By Lemma 3.1,

$$Z(3q) = q,$$

which is true by virtue of Lemma 4.2.15 in Majumdar [3]. The next prime is $p = 13$, so that $q = 13(k + 1) - 10$, $k \geq 1$. Some of the functions in this case are

$$Z(13\times 29) = 116,\ Z(13\times 107) = 428,\ Z(13\times 211) = 844.$$

Again, considering the prime $p = 23$, we get $q = 23(k + 1) - 10$, $k \geq 1$.

The first few functions in this case are

$$Z(23\times 59) = 413,\ Z(23\times 151) = 1057,\ Z(23\times 197) = 1379.$$

In Case 3, if $\mathrm{p} = 7$, then $\mathrm{q} = 7(\mathrm{k} + 1) - 10$, $\mathrm{k} \geq 1$. In this case, Lemma 3.1 gives

$$\mathrm{Z(7q) = 2q - 1},$$

which coincides with that given in Lemma 4.2.19 in Majumdar [3]. If $p = 17$, then q is of the form $q = 17(k + 1) - 10$, $k \geq 1$. The first few functions in this case are

$$Z(17\times 41) = 204,\ Z(17\times 109) = 544,\ Z(17\times 211) = 1054.$$

When $p = 37$, q is of the form $q = 37(k + 1) - 10$, $k \geq 1$. The first function is $Z(37\times 101) = 1110$, and the next one is $Z(37\times 397) = 4366$.

In Case 4, the first prime is $p = 19$ with $q = 19(k + 1) – 10, k \geq 1$. In this case, the first few functions are

$$Z(19\times 47) = 94, Z(19\times 199) = 398, Z(19\times 313) = 626.$$

The next prime is $p = 29$ with $q = 29(k + 1) – 10, k \geq 1$. The first such q is $q = 193$ (when $k = 6$) with $Z(29\times 193) = 579$. The next function is $Z(29\times 251) = 753$.

4. Conclusions

In this paper, we have derived the expression of *Z(pq)* with the help of pseudo Smarandache function *Z(n)*. Also we proved some related lemma of the expression *Z(pq)* with different values of *q*.

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