On the Closed Forms of $Z(pq)$, $q = kp \pm 10$

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Abstract

This paper derives the closed form expressions of $Z(pq)$, where $p$ and $q$ ($> p$) are primes, and $q$ is of the forms $q = kp \pm 10$ for some (positive) integer $k$, and $Z(.)$ is the pseudo Smarandache function.

Keywords: Pseudo Smarandache function, Diophantine equation.

1. Introduction

The pseudo Smarandache function $Z(n)$, introduced by Kashihara [1], is follows:

$$Z(n) = \min \{ m : n \mid \frac{m(m+1)}{2} \}.$$
The expressions of \( Z(n) \) for some particular cases of \( n \) are given by Kashihara [1], Asbacher [2] and Majumdar [3–5]. A method of finding \( Z(pq) \) is given in Theorem 4.2.2 in [3] (where \( p \) and \( q \ (> p) \) are primes) which is outlined below: Since

\[
Z(pq) = \min\{m : p q \mid \frac{m(m + 1)}{2}\},
\]

there are two possibilities:

Case 1. \( p \mid m, \ q \mid (m + 1) \).

In this case,

\[
m = px \text{ for some integer } x \geq 1,
\]

\[
m + 1 = qy \text{ for some integer } y \geq 1.
\]

This leads to the following Diophantine equation:

\[
qy - px = 1. \tag{1.1}
\]

Case 2. \( p \mid (m + 1), \ q \mid m \).

Here,

\[
m + 1 = px \text{ for some integer } x \geq 1,
\]

\[
m = qy \text{ for some integer } y \geq 1,
\]

so that the resulting Diophantine equation is

\[
px - qy = 1. \tag{1.2}
\]

Thus, the problem of finding \( Z(pq) \) reduces to the problem of solving the Diophantine equations (1.1) and (1.2) for minimum values of \( x \) or \( y \), as the case may be.

In [3], Majumdar gives the expressions of \( Z(pq) \), where \( p \) and \( q \ (> p) \) are primes and \( q \) is of the forms \( q = kp + \ell \) and \( q = (k + 1)p - \ell, \ 1 \leq \ell \leq 8 \). In this paper, we give an expression of \( Z(pq) \), when \( q \) is of the form \( q = kp + 10, \ k \geq 2 \) or \( q = (k + 1)p - 10, \)
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$k \geq 2$, to supplement the results in [3]. This is done in Section 2 and Section 3 respectively.

2. Values of $Z(pq), q = kp + 10$

In this section, we find an explicit form of $Z(pq)$, where $p$ and $q$ are primes with $q = kp + 10$ for some integer $k \geq 2$.

First note that, when $q = kp + 10$, $k \geq 2$, the Diophantine equations (1.1) and (1.2) become

$$(kp + 10)y - px = 1,$$

$$px - (kp + 10)y = 1,$$

that is,

$$10y - (x - ky)p = 1,$$

$$10y - (x - ky)p - 10y = 1.$$  \hspace{2cm} (2.1)

The lemma below gives the closed-form expression of $Z(pq)$, where $q = kp + 10$.

**Lemma 2.1.** Let $p$ and $q (> p)$ be two primes; moreover, let $q$ be of the form $q = kp + 10$ for some integer $k \geq 2$. Then,

$$Z(pq) = \begin{cases} 
\frac{q(p - 1)}{10}, & \text{if } 10|(p - 1) \\
\frac{q(3p + 1)}{10} - 1, & \text{if } 10|(p - 3) \\
\frac{q(3p - 1)}{10}, & \text{if } 10|(p - 7) \\
\frac{q(p + 1)}{10} - 1, & \text{if } 10|(p - 9)
\end{cases}$$
**Proof.** We consider the following four cases that may arise.

Case 1. \( p = 10a + 1 \) for some integer \( a \geq 1 \).

In this case, the Diophantine equations (2.1) and (2.2) read as

\[
1 = 10y - (x - ky)(10a + 1) = 10(y - (x - ky)a) - (x - ky),
\]

\[
1 = (x - ky)(10a + 1) - 10y = (x - ky) - 10(y - (x - ky)a).
\]

Clearly, the minimum solution is obtained from the second of the above two equations with

\[
x - ky = 1, \ y - (x - ky)a = 0.
\]

Then, the minimum solution is \( y = a \), and hence, the minimum \( m \) is given by

\[
m = qy = \frac{q(p - 1)}{10}.
\]

Case 2. \( p = 10a + 3 \) for some integer \( a \geq 0 \).

Here, (2.1) and (2.2) become

\[
1 = 10y - (x - ky)(10a + 3) = 10(y - (x - ky)a) - 3(x - ky),
\]

\[
1 = (x - ky)(10a + 3) - 10y = 3(x - ky) - 10(y - (x - ky)a).
\]

The minimum solution is obtained from the first of the above two Diophantine equations with

\[
x - ky = 3, \ y - (x - ky)a = 1.
\]

Therefore, the minimum \( y \) is \( y = 3a + 1 \), and consequently, the minimum \( m \) is

\[
m = qy - 1 = q(3a + 1) - 1 = \frac{q(3p + 1)}{10} - 1.
\]

Case 3. \( p = 10a + 7 \) for some integer \( a \geq 0 \).

Here, the Diophantine equations satisfied are

\[
1 = 10y - (x - ky)(10a + 7) = 10(y - (x - ky)a) - 7(x - ky),
\]

\[
1 = (x - ky)(10a + 7) - 10y = 7(x - ky) - 10(y - (x - ky)a).
\]
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1 = (x - ky)(10a + 7) - 10y = 7(x - ky) - 10[y - (x - ky)a],

For which the minimum solution is obtained from the second equation as

\[ x - ky = 3, \quad y - (x - ky)a = 2. \]

Thus, the minimum solution is $y = 3a + 2$, and the minimum $m$ is

\[ m = qy = q(3a + 2) = \frac{q(3p - 1)}{10}. \]

Case 4. $p = 10a + 9$ for some integer $a \geq 1$.

In this case, the Diophantine equations (2.1) and (2.2) take the forms

\[ 1 = 10y - (x - ky)(10a + 9) = 10[y - (x - ky)a] - 9(x - ky), \]

\[ 1 = (x - ky)(10a + 9) - 10y = 9(x - ky) - 10[y - (x - ky)a]. \]

Clearly, the minimum solution is obtained from the first equation, with

\[ x - ky = 1, \quad y - (x - ky)a = 1. \]

Thus, $y = a + 1$, and the minimum $m$ is given by

\[ m = qy - 1 = q(a + 1) - 1 = \frac{q(p + 1)}{10} - 1. \]

All these complete the proof of the lemma.

We now give some examples of the application of Lemma 2.1. In Case 1 in the proof of the lemma, letting $a = 1$, we get the prime $p = 11$, so that the prime $q$ is of the form $q = 11k + 10, \quad k \geq 1$. The first few functions in this case are

\[ Z(11 \times 43) = 43, \quad Z(11 \times 109) = 109, \quad Z(11 \times 131) = 131. \]

The next prime is $p = 31$, which gives, for example, $Z(31 \times 41) = 123$ and $Z(31 \times 103) = 309$.

In Case 2, $a = 0$ gives the prime $p = 3$, and so, $q = 3k + 10, \quad k \geq 1$. Lemma 2.1 gives

\[ Z(3q) = q - 1, \]
which is true by Lemma 4.2.15 of Majumdar [3]. Thus, Case 2 holds true for \( a = 0 \) as well. The next prime is \( p = 13 \), and we get, for example,
\[
Z(13 \times 23) = 91, \quad Z(13 \times 101) = 403, \quad Z(13 \times 127) = 507.
\]
In Case 3, if \( p = 7 \), then \( q = 7k + 10, \ k \geq 1 \), and in this case, Lemma 2.1 gives
\[
Z(7q) = 2q,
\]
which is validated by Lemma 4.2.19 of Majumdar [3]. Thus, Case 3 holds for \( a = 0 \) as well. The next prime in the sequence is \( p = 17 \), so that \( q = 17k + 10, \ k \geq 1 \). The first few functions are
\[
Z(17 \times 61) = 305, \quad Z(17 \times 163) = 815, \quad Z(17 \times 197) = 985.
\]
In Case 4, the first prime is \( p = 19 \), so that \( q = 19k + 10, \ k \geq 1 \). Corresponding to this case, we get the following functions:
\[
Z(19 \times 29) = 57, \quad Z(19 \times 67) = 133, \quad Z(19 \times 181) = 361.
\]
The next prime is \( p = 29 \), so that \( q = 29k + 10, \ k \geq 1 \). Some of the functions in this case are
\[
Z(29 \times 97) = 290, \quad Z(29 \times 271) = 812, \quad Z(29 \times 503) = 1508.
\]

3. Values of \( Z(pq), \ q = (k + 1)p - 10 \)

This section derives an expression of \( Z(pq) \), where \( p \) and \( q \) are primes, and \( q \) is of the form \( q = (k + 1)p - 10 \) for some integer \( k \geq 2 \).
When \( q = (k + 1)p - 10 \), the Diophantine equations (1.1) and (1.2) become
\[
\begin{align*}
((k + 1)p - 10)y - px &= 1, \\
px - [(k + 1)p - 10]y &= 1,
\end{align*}
\]
that is,

\[ I = ((k + 1)y - x)p - 10y, \quad (3.1) \]
\[ I = 10y - ((k + 1)y - x)p. \quad (3.2) \]

We now prove the following lemma, giving an expression of \( Z(pq) \) with \( q = (k + 1)p - 10 \).

**Lemma 3.1.** Let \( p \) and \( q \) (\( > p \)) be two primes with \( q = (k + 1)p - 10 \) for some integer \( k \geq 2 \). Then,

\[ Z(pq) = \begin{cases} 
\frac{q(p - 1)}{10} - 1, & \text{if } 10 | (p - 1) \\
\frac{q(3p + 1)}{10}, & \text{if } 10 | (p - 3) \\
\frac{q(3p - 1)}{10} - 1, & \text{if } 10 | (p - 7) \\
\frac{q(p + 1)}{10}, & \text{if } 10 | (p - 9) 
\end{cases} \]

**Proof.** We consider below separately the four cases that may arise:

**Case 1.** \( p = 10a + 1 \) for some integer \( a \geq 1 \).

In this case, the Diophantine equations (3.1) and (3.2) may be rewritten as

\[ I = [(k + 1)y - x](10a + 1) - 10y = [(k + 1)y - x] - 10[y - ((k + 1)y - x)a], \]
\[ I = 10y - [(k + 1)y - x](10a + 1) = 10[y - ((k + 1)y - x)a] - [(k + 1)y - x]. \]

The first of these two give the minimum solution, namely,

\( (k + 1)y - x = 1, y - ((k + 1)y - x)a = 0. \)

The minimum solution is thus \( y = a \), and consequently, the minimum \( m \) is

\[ m = qy - 1 = \frac{q(p - 1)}{10} - 1. \]

**Case 2.** \( p = 10a + 3 \) for some integer \( a \geq 0 \).
Here, (3.1) and (3.2) become

\[ 1 = [(k + 1)y - x](10a + 3) - 10y = 3[(k + 1)y - x] - 10[y - (k + 1)y - x]a, \]
\[ 1 = 10y - [(k + 1)y - x](10a + 3) = 10[y - (k + 1)y - x]a - 3[(k + 1)y - x]. \]

The minimum solution, obtained from the second of the above two equations, is

\[ (k + 1)y - x = 3, \quad y - (k + 1)y - x]a = 1. \]

Then, the minimum solution is \( y = 3a + 1 \), and the minimum \( m \) is

\[ m = qy = \frac{q(3p + 1)}{10}. \]

Case 3. \( p = 10a + 7 \) for some integer \( a \geq 0 \).

In this case, from (3.1) and (3.2), we have

\[ 1 = [(k + 1)y - x](10a + 7) - 10y = 7[(k + 1)y - x] - 10[y - (k + 1)y - x]a, \]
\[ 1 = 10y - [(k + 1)y - x](10a + 7) = 10[y - (k + 1)y - x]a - 7[(k + 1)y - x]. \]

The minimum solution is then obtained from the first equation as follows:

\[ (k + 1)y - x = 3, \quad y - (k + 1)y - x]a = 2. \]

Thus, \( y = 3a + 2 \), and the minimum \( m \) is

\[ m = qy - 1 = \frac{q(3p - 1)}{10} - 1. \]

Case 4. \( p = 10a + 9 \) for some integer \( a \geq 1 \).

From the Diophantine equations (3.1) and (3.2), we have

\[ 1 = [(k + 1)y - x](10a + 9) - 10y = 9[(k + 1)y - x] - 10[y - (k + 1)y - x]a, \]
\[ 1 = 10y - [(k + 1)y - x](10a + 9) = 10[y - (k + 1)y - x]a - 9[(k + 1)y - x]. \]

Clearly, the minimum solution is obtained from the second equation, which is

\[ (k + 1)y - x = 1, \quad y - (k + 1)y - x]a = 1. \]

This gives the minimum solution \( y = a + 1 \), and the minimum \( m \) is
In Case 1, the first prime is \( p = 11 \) with \( q = 11(k + 1) - 10, \ k \geq 1 \). Some of the functions are
\[
Z(11 \times 23) = 22, \ Z(11 \times 67) = 66, \ Z(11 \times 89) = 88.
\]
The next prime of the sequence is \( p = 31 \), so that \( q = 31(k + 1) - 10, \ k \geq 1 \). The first such \( q \) is \( q = 83 \), with \( Z(31 \times 83) = 248 \). The next function is \( Z(31 \times 269) = 806 \).

In Case 2, the first prime is \( p = 3 \) with \( q = 3(k + 1) - 10, \ k \geq 1 \). By Lemma 3.1,
\[ Z(3q) = q, \]
which is true by virtue of Lemma 4.2.15 in Majumdar [3]. The next prime is \( p = 13 \), so that \( q = 13(k + 1) - 10, \ k \geq 1 \). Some of the functions in this case are
\[
Z(13 \times 29) = 116, \ Z(13 \times 107) = 428, \ Z(13 \times 211) = 844.
\]
Again, considering the prime \( p = 23 \), we get \( q = 23(k + 1) - 10, \ k \geq 1 \).

The first few functions in this case are
\[
Z(23 \times 59) = 413, \ Z(23 \times 151) = 1057, \ Z(23 \times 197) = 1379.
\]
In Case 3, if \( p = 7 \), then \( q = 7(k + 1) - 10, \ k \geq 1 \). In this case, Lemma 3.1 gives
\[ Z(7q) = 2q - 1, \]
which coincides with that given in Lemma 4.2.19 in Majumdar [3]. If \( p = 17 \), then \( q \) is of the form \( q = 17(k + 1) - 10, \ k \geq 1 \). The first few functions in this case are
\[
Z(17 \times 41) = 204, \ Z(17 \times 109) = 544, \ Z(17 \times 211) = 1054.
\]
When \( p = 37 \), \( q \) is of the form \( q = 37(k + 1) - 10, \ k \geq 1 \). The first function is \( Z(37 \times 101) = 1110 \), and the next one is \( Z(37 \times 397) = 4366 \).
In Case 4, the first prime is \( p = 19 \) with \( q = 19(k + 1) - 10, \ k \geq 1 \). In this case, the first few functions are
\[
Z(19 \times 47) = 94, \ Z(19 \times 199) = 398, \ Z(19 \times 313) = 626.
\]
The next prime is \( p = 29 \) with \( q = 29(k + 1) - 10, \ k \geq 1 \). The first such \( q \) is \( q = 193 \) (when \( k = 6 \)) with \( Z(29 \times 193) = 579 \). The next function is \( Z(29 \times 251) = 753 \).

4. Conclusions

In this paper, we have derived the expression of \( Z(pq) \) with the help of pseudo Smarandache function \( Z(n) \). Also we proved some related lemma of the expression \( Z(pq) \) with different values of \( q \).

References


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