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An Exact Solution of the Reaction-Diffusion Equation for the Speed of the Interface Propagation in Superconductors

Neelufar Panna

Department of Physics, University of Chittagong.Chittagong-4331. Bangladesh
E-mail: neelcu@yahoo.com

Abstract

The speed of interface propagation in superconductors for the scalar reaction-diffusion equation $u_t = \nabla^2 u + F(u)$ is studied in detail. Here the non linear reaction term $F(u)$ is the time-dependent Ginzburg-Landau or TDGL equation $F(u) = u - u^3$ which describes the dynamics of the order-disorder transition. In contrast to what has been done in previous work [1] here an improved exact solution has derived by using TDGL equation to determine the speed of the front propagation. The analytical treatment of this study has been found in good agreement with the numerical simulation of V. Mendez et al. [2] and Di Bartolo and Dorsey [3].

***Keywords**: Front propagation in superconductor; Ginzburg-Landau equation; velocity selection; exact solution.*

AwZcwievnx‡Z Avšÿcô wе¯ÍiY Gi `ªZ wе¯ÍwiZ Av‡jvPbv Kiv n‡q‡Q †¯লার বিক্রিয়া-wemiY mgxKiY $u_t = \nabla^2 u + F(u)$ Gi Rb¨| GLv‡b A‰iwLK বিক্রিয়v c` $F(u)$ n‡"Q mgqwbf©i Ginzburg-Landau A_ev TDGL mgxKiY $F(u) = u - u^3$, ‡hwU kšLjv-wekšLjv Ae¯'všÍ‡ii MwZwe`¨v eY©bv K‡i| Avšÿcô wе¯Íi‡bi `ªZ wbY©‡qi Rb¨ cee©ZxKv‡Ri [1]

wecix‡Z Ab¨ GKwU h_vh_ mgvavb †ei Kiv n‡q‡Q TDGL mgxKiY e¨envi K‡i| GB Kv‡Ri we‡kølY eY©bv V. Méndez et al. [2] Ges Di Bartolo I Dorsey [3] Gi msL¨vZwZ¡K djvd‡ji mv‡_ msMwZc~Y©|

1. Introduction

The study of interface propagation in superconductor is one of the most fundamental problems in non equilibrium physics. In these problems the system is first prepared carefully in an unstablestate by preparing an experimental system in a state it does not naturally stay in. This occurs when a sudden destabilizing change is applied, a system responds by forming fronts which propagates into the unstable states.

In general the nonlinear equation have been employed to model front propagation in different areas such as dendrites and population growth, pulse propagation in nerves, and many other biological, chemical and physical phenomena are described by reaction diffusion equations which are of the form

$$u_t(\mathbf{r},t) = \nabla^2 u(\mathbf{r},t) + F(u(\mathbf{r},t)) \qquad (1.1)$$

Here $u(\mathbf{r},t)$ is a field variable (e.g. order parameter, populationdensity,magnetization chemical concentration) defined as a function of space (**r**) and time (t).The nonlinearfunction $F(u(\mathbf{r},t))$ is a reaction term. There are two cases. One is the Fisherequation $F(u) = u - u^2$, which describes the dynamics of structured population [4]. Another one is the time-dependent Ginzburg-Landau or TDGL equation, $F(u) = u - u^3$ describes the dynamics of the order-disorder transition [5].

Now particular term of equation (1.1)is the Fisher equation

$$\frac{\partial u(\mathbf{r},t)}{\partial t} = \nabla^2 u(\mathbf{r},t) + u(\mathbf{r},t)[1-u(\mathbf{r},t)] \qquad (1.2)$$

For one dimensional (1D) case Kolmogorov, Petrovskii and Piscounov (KPP) [6] wrote equation (1.2) in the form as

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t)[1-u(x,t)] \qquad (1.3)$$

According to KPP equation (1.3) has stable traveling wave solution (they called 'clines') which are walls traveling in the +x direction with velocity $v \geq 2$ and $u(-\infty,\mathrm{t}) = 1$, $u(\infty,t) = 0$; or walls traveling in the $-\mathrm{x}$ direction with velocity $v \leq 2$ and $u(-\infty,\mathrm{t}) = 0$, $u(\infty,\mathrm{t}) = 1$, but whose analytical result were not known.

Subsequently, Aronson and Weinberger [7], demonstrated the powerful result by considering the 1D version of equation (1.1),

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(xt)}{\partial x^2} + F(u(x,t)) \qquad (1.4)$$

Where $F(x) \geq 0, F(0) = F(1) = 0; \quad F'(0) > 0 > F'(1)$

They found that for a sufficiently broad class of initial condition the solutions of this equation evolve into fronts with a definite speed *v*, which satisfies

$2\sqrt{F'(0)} \le v \le 2\sup\sqrt{F(u)/u}$ so that for the special case $F(u) = u - u^3$ the selected speed is *v*=2.

According to the work of Di Bartolo and Dorsy [3] consider a sample of superconducting material embedded in a stationary applied magnetic field H equal to the critical field H_c so that there is a stationary planar superconducting-normal interface grows which separates the normal and superconducting phases. After than the magnetic field is rapidly removed, the interface becomes dynamically unstable and propagates towards the normal phase so as to expel any trapped magnetic flux, leaving the sample in Meissner state. It has been considered that the interface remains plannar during all the processes.

Several methods are proposed for the analysis of dynamical velocity selection for fronts such as marginal stability hypothesis, variational speed selection [8], structural stability and construction of exact solutions [1] etc. In this work the front propagation speed is determined by using the construction of exact solution method.

The purpose of this paper is to present a detailed study of the dynamics of the front propagation and hence to find out the front velocity in superconductor. This is an important problem to be solved and is an interesting issue to the Physics society. In section 2 construction of exact solution of the time dependent Ginzberg-Landau (TDGL) equations for superconducting fronts are presented. The starting points are TDGL equations which provide the self consistent description of the coupling between the order parameter and the vector potential. Section 3 gives the results and discussions.

2. Theory

To study the behavior of the superconducting-normal interface, TDGL equations in dimensionless unit are introduced in one dimension as:

$$\partial_t f = \frac{1}{\kappa^2} \partial_x^2 f - q^2 f + f - f^3 \tag{2.1}$$

$$\overline{\sigma}\, \partial_t q = \partial_x^2 q - f^2 q \tag{2.2}$$

Here f is the magnitude of the order parameter ψ of the superconductor, q is the vector potential which is gauge invariant and related to the magnetic field as $h = \partial_x q$, $\overline{\sigma}$ is the dimensionless normal state conductivity which is the ratio of the order parameter diffusion constant $D_\psi = \hbar/2m\gamma$ (γ is the order parameter relaxation time, m is the mass of Cooper pair) to the magnetic field diffusion constant $D_h = 1/4\pi\sigma^{(n)}$ and κ is the Ginzberg-Landau parameter.

The interesting case is finding traveling wave solutions for the model discussed in this paper. The steady traveling wave solution for the TDGL equation can be written in the form as $f(x,t) = F(X) = F(x - vt)$ and $q(x,t) = Q(X) = Q(x - vt)$, where $X = x - vt$ with $v > 0$. Then equations (2.1) and (2.2) become

$$\frac{1}{\kappa^2}F'' + vF' - Q^2F + F - F^3 = 0 \tag{2.3}$$

$$Q'' + \bar{\sigma}vQ' - F^2Q = 0 \tag{2.4}$$

To determine front velocity as a function of $\bar{\sigma}$, a modification is given to the interface problem. Let us consider the following generalized Ginzberg-Landau equations

$$\left(1+\beta^2\right)F'' + vF' - Q^2F + F - F^3 = 0 \tag{2.5}$$

$$Q'' - \sigma_0 Q' - Q\left(b_1F + b_2F^2 + b_3F^3\right) = 0 \tag{2.6}$$

Here b_1, b_2, b_3 are constant parameters and $\sigma_0 = \bar{\sigma}v$.

The exact solution for order parameter and vector potential can e constructed as

$$F = \frac{\lambda_1 + \lambda_2 e^{\xi X} + \lambda_3 e^{2\xi X}}{1 + \alpha_1 e^{\xi X} + \alpha_2 e^{2\xi X}} \tag{2.7}$$

$$Q = \frac{\alpha\beta e^{\xi X} + Ce^{2\xi X}}{1 + \alpha_1 e^{\xi X} + \alpha_2 e^{2\xi X}} \tag{2.8}$$

Where $\alpha, \alpha_1, \alpha_2, \lambda_1$ and λ_2 are any arbitrary constants in front solution. Now

$$Q' = \frac{\alpha\beta\xi e^{\xi X} + 2\xi Ce^{2\xi X}}{(1 + \alpha_1 e^{\xi X} + \alpha_2 e^{2\xi X})} - \frac{\left(\alpha\beta e^{\xi X} + Ce^{2\xi X}\right)\left(\alpha_1\xi e^{\xi X} + 2\xi\alpha_2 e^{2\xi X}\right)}{\left(1 + \alpha_1 e^{\xi X} + \alpha_2 e^{2\xi X}\right)^2} \tag{2.9}$$

and

$$Q'' = \frac{\alpha\beta\xi^2 e^{\xi X} + 4C\xi^2 e^{2\xi X}}{(1+\alpha_1 e^{\xi X} + \alpha_2 e^{2\xi X})} - \frac{(\alpha\beta\xi e^{\xi X} + 2C\xi e^{2\xi X})(\alpha_1\xi e^{\xi X} + 2\xi\alpha_2 e^{2\xi X})}{(1+\alpha_1 e^{\xi X} + \alpha_2 e^{2\xi X})^2} - \frac{(\alpha\beta\xi e^{\xi X} + 2C\xi e^{2\xi X})(\alpha_1\xi e^{\xi X} + 2\xi\alpha_2 e^{2\xi X}) + (\alpha\beta e^{\xi X} + C\xi e^{2\xi X})(\alpha_1\xi^2 e^{\xi X} + 4\xi^2\alpha_2 e^{2\xi X})}{(1+\alpha_1 e^{\xi X} + \alpha_2 e^{2\xi X})^2} + \frac{2(\alpha\beta e^{\xi X} + Ce^{2\xi X})(\alpha_1\xi e^{\xi X} + 2\xi\alpha_2 e^{2\xi X})^2}{(1+\alpha_1 e^{\xi X} + \alpha_2 e^{2\xi X})^3} \tag{2.10}$$

Substituting equations (2.7- 2.10) into equation (2.6), we get

$$\frac{\alpha\beta\xi^2 e^{\xi X} + 4C\xi^2 e^{2\xi X}}{(1+\alpha_1 e^{\xi X} + \alpha_2 e^{2\xi X})} - \frac{(\alpha\beta\xi e^{\xi X} + 2C\xi e^{2\xi X})(\alpha_1\xi e^{\xi X} + 2\xi\alpha_2 e^{2\xi X})}{(1+\alpha_1 e^{\xi X} + \alpha_2 e^{2\xi X})^2} - \frac{(\alpha\beta\xi e^{\xi X} + 2C\xi e^{2\xi X})(\alpha_1\xi e^{\xi X} + 2\xi\alpha_2 e^{2\xi X}) + (\alpha\beta e^{\xi X} + C\xi e^{2\xi X})(\alpha_1\xi^2 e^{\xi X} + 4\xi^2\alpha_2 e^{2\xi X})}{(1+\alpha_1 e^{\xi X} + \alpha_2 e^{2\xi X})^2} + \frac{2(\alpha\beta e^{\xi X} + Ce^{2\xi X})(\alpha_1\xi e^{\xi X} + 2\xi\alpha_2 e^{2\xi X})^2}{(1+\alpha_1 e^{\xi X} + \alpha_2 e^{2\xi X})^3} - \sigma_0\left[\frac{\alpha\beta\xi e^{\xi X} + 2C\xi e^{2\xi X}}{(1+\alpha_1 e^{\xi X} + \alpha_2 e^{2\xi X})} - \frac{(\alpha\beta\xi e^{\xi X} + C\xi e^{2\xi X})(\alpha_1\xi e^{\xi X} + 2\xi\alpha_2 e^{2\xi X})}{(1+\alpha_1 e^{\xi X} + \alpha_2 e^{2\xi X})^2}\right] - \frac{\alpha\beta e^{\xi X} + Ce^{2\xi X}}{(1+\alpha_1 e^{\xi X} + \alpha_2 e^{2\xi X})}\left[b_1\frac{\lambda_1 + \lambda_2 e^{\xi X} + \lambda_3 e^{2\xi X}}{1+\alpha_1 e^{\xi X} + \alpha_2 e^{2\xi X}} + b_2\frac{(\lambda_1 + \lambda_2 e^{\xi X} + \lambda_3 e^{2\xi X})^2}{(1+\alpha_1 e^{\xi X} + \alpha_2 e^{2\xi X})^2}\right] = 0 \tag{2.11}$$

Let $b_3 = 0$. Terms with $(1+\alpha_1 e^{\xi X} + \alpha_2 e^{2\xi X})^{-1}$ from equation (2.11) implies:

$$\frac{\alpha\beta\xi^2 e^{\xi X} + 4C\xi^2 e^{2\xi X}}{(1+\alpha_1 e^{\xi X} + \alpha_2 e^{2\xi X})} - \sigma_0\frac{\alpha\beta\xi e^{\xi X} + 2C\xi e^{2\xi X}}{(1+\alpha_1 e^{\xi X} + \alpha_2 e^{2\xi X})} = 0 \tag{2.12}$$

$$\alpha\beta\xi^2 e^{\xi X} + 4C\xi^2 e^{2\xi X} - \sigma_0(\alpha\beta\xi e^{\xi X} + 2C\xi e^{2\xi X}) = 0$$

Equating the coefficients of $e^{\xi X}$, we get

$$\alpha\beta\xi^2 - \sigma_0\alpha\beta\xi = 0$$

$$\sigma_0 = \xi$$

For $\xi = \frac{1}{\sqrt{2}}$ this can be written as

$$\bar{\sigma} v = \frac{1}{\sqrt{2}} \because \sigma_0 = \bar{\sigma} v$$

Hence $$v = \frac{1}{(\sqrt{2})\bar{\sigma}} \tag{2.13}$$

The front speed in terms of dimensionless normal state conductivity which determines the rate at which flux diffuses in the normal state is obtained. Note that as $\bar{\sigma}$ increases v decreases that means the larger the flux diffuses in the front the smaller the front speed.

Di Bartolo and Dorsey [3] constructed an exact solution for F and Q. For the perturbed approximation of equations (2.3) and (2.4), these solutions are used as the starting point to calculate the front speed. The selected velocity for superconducting-normal interface is

$$v = \frac{1}{(\sqrt{2})\bar{\sigma} Q_\infty + \beta_k} \tag{2.14}$$

Where β_k is the kinetic coefficient and Q_∞ is the integrated magnetic field in the front which is an important control parameter for the front dynamics i.e., the larger the trapped magnetic flux in the front the smaller the front speeds. We see that for $Q_\infty = 1$ and $\beta_k = 0$, equation (2.14) is in excellent agreement with our result equation (2.13).

3. Results and Discussion

We know fronts propagate into an unstable state in superconductor with a continuous order parameter at a unique shape and speed. There already exist several proposed criteria for the analysis of the dynamical velocity selection. In this paper exact solution method has been applied to TDGL equation in order to study the front speed.

Figure 1 represents the graph for the speed of the fronts for different values of $\bar{\sigma}$ according to the analytical expression, equation (2.13), $v = \frac{1}{(\sqrt{2})\bar{\sigma}}$ .It has been shown that when $\bar{\sigma} = \frac{1}{2}, \beta_k = 0$ and $Q_\infty = 1$,equation(2.14) yields v=1.5975, while following equation (2.13),the result yields v=1.414 which coincides with the numerical result using equation (2.1) and equation (2.2) of Di Bartolo [3].This is also in agreement with the numerical simulation for the dimensionless speed as a function of the dimensionless parameter ‘a’ of V. Méndez et al.[2] in their front solutions to hyperbolic reaction-diffusion equations. The result coincides with the superconducting-normal interface propagation speed as a function of G-L parameter performed by A.de la Cruz [8] using variational method.

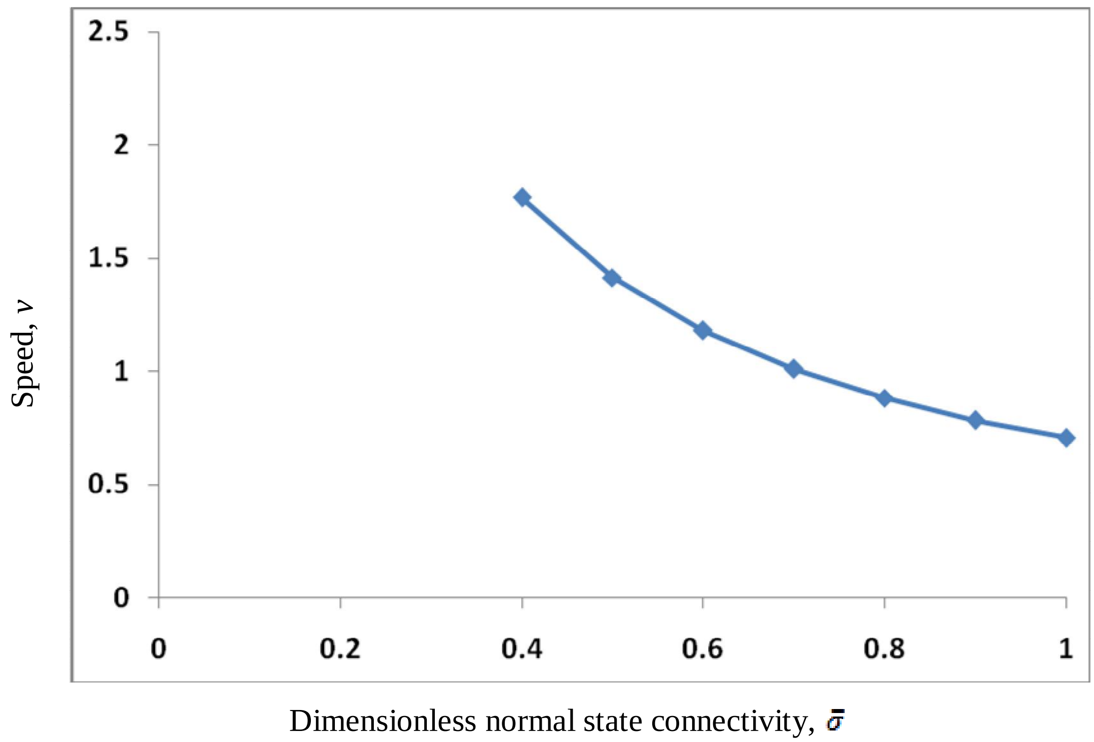

Fig-1: The speed v of the front, obtained from the equation (2.13) is plotted as a function of dimensionless normal state conductivity $\bar{\sigma}$.

4. Conclusions

Throughout this work, the propagation of fronts separating the superconducting and normal phases, which are produced after a quench to the zero applied magnetic field have performed by a new approach. Based on this approach I have obtained an expression for the superconducting – normal interface propagation speed. In my future work I will try to use two reaction-diffusion equations (parabolic and hyperbolic) from the variational point of view to obtain the selected speed of superconducting fronts exactly.

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