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(U, M) -DERIVATIONS IN PRIME Γ-RINGS

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Abstract

In this article, we define (*U*,*M*)-derivation *d* of a Γ -ring M . For a Lie ideal *U* of a 2 -torsion free prime Γ -ring *M* satisfying the condition $a\alpha b\beta c = a\beta b\alpha c$ for all $a,b,c \in M$ and $\alpha, \beta \in \Gamma$, we prove the following results:

(i) if U is an admissible Lie ideal of *M*, then $d(u\alpha v) = d(u)\alpha v + u\alpha d(v)$ for all $u,v \in U$, $\alpha \in \Gamma$

(ii) if $u\alpha u \in U$ for all $u \in U, \alpha \in \Gamma$, then $d(u\alpha m) = d(u)\alpha m + u\alpha d(m)$ for all $m \in M$.

Keywords : Square closed Lie ideal, admissible Lie ideal, (*U*,*M*)-derivation, prime Γ -ring.

Introduction

Herstein (1957) proved a well-known result in prime rings which states that every Jordan derivation is a derivation. Afterwards many mathematicians studied extensively the derivations in prime rings. Awtar (1984) extended this result to Lie ideals. (*U*,*R*)-derivations in rings have been introduced by Faraj, Haetinger and Majeed (2010), as a generalization of Jordan derivation on Lie ideals of a ring. The notion of (*U*,*R*)-derivation extends the concept given by Awtar (1984). Faraj, Haetinger and Majeed (2010) proved that if R is a prime ring, char$(R) \neq 2$, U is a square closed Lie ideal of *R* and *d* is a (*U*,*R*)-derivation of R , then $d(ur) = d(u)r + ud(r), u \in U, r \in R$.

This result is a generalization of a result of Awtar (1984). The notion of a Γ -ring has been developed by Nobusawa (1964), as a generalization of a ring. Barnes (1966) generalized the concept of Nobusawa's Γ -ring. Nowadays, Γ -ring theory is a showpiece of mathematical unification, bringing together several branches of the subject. It is an active research area for mathematicians and in the last 40 years, many classical ring theoritical concepts and results have been generalized to Γ -rings by many authors. The notions of derivation and Jordan derivation in Γ -rings have been introduced by Sapanci and Nakajima (1997). More recently,

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some extensive results of left derivation and Jordan left derivation of a Γ -ring were determined by Ceven (2002). Halder and Paul (2012) extended the results of Ceven (2002) to Lie ideals. in the light of some significant results due to Jordan left derivation of a classical ring obtained by Jun and Kim (1996),

In this article, we introduce the concept of (*U*,*M*)-derivation, where U is a Lie ideal of a Γ-ring M. An example of a Lie ideal of a Γ-ring and an example of a (*U*,*M*)-derivation are given here. A result of Faraj, Haetinger and Majeed (2010) is generalized in Γ-rings by the new concept of (*U*,*M*)-derivation.

A Γ-ring M is called a *prime* Γ*-ring* if for all $a,b \in M$, $a\Gamma M\Gamma b = 0$ implies $a = 0$ or $b = 0$ and M is called *semiprime* if $a\Gamma M\Gamma a = 0$ (with $a \in M$) implies $a = 0$. A Γ-ring M is *2-torsion free* if $2a = 0$ implies $a = 0$ for all $a \in M$. For any $x, y \in M$ and $\alpha \in \Gamma$, we define the *Lie product* by $[x, y]_\alpha = x\alpha y - y\alpha x$. An additive subgroup $U \subseteq M$ is said to be a *Lie ideal* of M if whenever $u \in U, m \in M$ and $\alpha \in \Gamma$, then $[u,m]_\alpha \in U$. In the main results of this article we assume that the Lie ideal U satisfies $u\alpha u \in U$ for all $u \in U$. A Lie ideal of this type is called a *square closed Lie ideal*. Furthermore, if the Lie ideal U is square closed and U is not contained in $Z(M)$, where $Z(M)$ denotes the center of M, then U is called an *admissible Lie ideal* of M. Throughout the article, we ausme that $a\alpha b\beta c = a\beta b\alpha c$ holds for all $a,b,c \in M$ and $\alpha, \beta \in \Gamma$, we refer to this condition by (*).

We note the basic commutator identities. $[x\alpha y, z]_\beta = [x, z]_\beta \alpha y + x[\alpha, \beta]_z y + x\alpha[y, z]_\beta$, and $[x, y\alpha z]_\beta = [x, y]_\beta \alpha z + y[\alpha, \beta]_x z + y\alpha[x, z]_\beta$ for all $a,b,c \in M$ and $\alpha, \beta \in \Gamma$. According to the condition (*), the above two identities reduces to $[x\alpha y, z]_\beta = [x, z]_\beta \alpha y + x\alpha[y, z]_\beta$, and $[x, y\alpha z]_\beta = [x, y]_\beta \alpha z + y\alpha[x, z]_\beta$ for all $a,b,c \in M$ and $\alpha, \beta \in \Gamma$.

Preliminaries

Definition 1. *Suppose U is a Lie ideal of a Γ-ring M. An additive mapping $d : M \to M$ is a* (*U*,*M*) - derivation *of M if $d(u\alpha m + s\alpha u) = d(u)\alpha m + u\alpha d(m) + d(s)\alpha u + s\alpha d(u)$ holds for all $u \in U; m, s \in M$ and $\alpha \in \Gamma$.*

The existence of a Lie ideal of a Γ-ring and a (*U*,*M*) - derivation of a Γ-ring are confirmed by the following example.

Example 1. *If R is an associative ring with 1, and U is a Lie ideal of R. Let $M = M_{1,2}(R)$*

and $\Gamma = \left\{ \begin{pmatrix} n.1 \\ 0 \\ \end{pmatrix} : n \in Z \right\}$*, then* M *is a* Γ*-ring. Let* $N = \{(x,x) : x \in R\} \subseteq M$ *, then* N *is a sub* Γ*-ring. Let* $U_1 = \{(u,u) : u \in U\}$*, then for* $una - anu \in U$

$$(u,u)\begin{pmatrix} n \\ 0 \\ \end{pmatrix}(a,a) - (a,a)\begin{pmatrix} n \\ 0 \\ \end{pmatrix}(u,u) = (una,una) - (anu,anu)$$

$$= (una - anu, una - anu) \in U.$$

Thus, U_1 is a Lie ideal of N. Let $d : R \to R$ be a (U,R)-derivation. Now, we define a mapping $D : N \to N$ by $D((x,x)) = (d(x), d(x)).$ Then

$$D((u,u)\begin{pmatrix} n \\ 0 \end{pmatrix}(a,a) + (b,b)\begin{pmatrix} n \\ 0 \end{pmatrix}(u,u)) = D((una,una) + (bnu,bnu))$$
$$= D((una + bnu, una + bnu)) = (d(una + bnu), d(una + bnu))$$
$$= (d(u)na + und(a) + d(b)nu + bnd(u), d(u)na + und(a) + d(b)nu + bnd(u))$$
$$= (d(u)na + und(a), d(u)na + und(a)) + (d(b)nu + bnd(u), d(b)nu + bnd(u))$$
$$= (d(u)na, d(u)na) + (und(a), und(a)) + (d(b)nu, d(b)nu) + (bnd(u), bnd(u))$$

$$= (d(u),d(u))\begin{pmatrix} n \\ 0 \end{pmatrix}(a,a) + (u,u)\begin{pmatrix} n \\ 0 \end{pmatrix}(d(a),d(a)) + (d(b),d(b))\begin{pmatrix} n \\ 0 \end{pmatrix}(u,u) + (b,b)\begin{pmatrix} n \\ 0 \end{pmatrix}(d(u),d(u))$$

$$= D((u,u))\begin{pmatrix} n \\ 0 \end{pmatrix}(a,a) + (u,u)\begin{pmatrix} n \\ 0 \end{pmatrix}(D((a,a)) + D((b,b))\begin{pmatrix} n \\ 0 \end{pmatrix}(u,u) + (b,b)\begin{pmatrix} n \\ 0 \end{pmatrix}D((u,u))$$
$$= D(u_1)\alpha x + u_1 \alpha D(x) + D(y)\alpha u_1 + y\alpha D(u_1),$$

where $u_1 = (u,u), \alpha = \begin{pmatrix} n \\ 0 \end{pmatrix}, x = (a,a), y = (b,b).$

Therefore, $D(u_1 \alpha x + y \alpha u_1) = D(u_1)\alpha x + u_1 \alpha D(x) + D(y)\alpha u_1 + y\alpha D(u_1).$

Hence D is a $(U_1, N)-$derivation of N.

(*U*,*M*)-derivations in Prime Γ-Rings

In order to prove the main results, we prove some preparatory results concerning (*U*,*M*)-derivations of a Γ-ring, and list them as lemmas.

Lemma 1. *Let* d *be a* (*U*,*M*)-derivation *of* M . *Then*

(*i*) $d(u\alpha m\beta u) = d(u)\alpha m\beta u + u\alpha d(m)\beta u + u\alpha m\beta d(u)$ for all $u \in U, m \in M$ and $\alpha, \beta \in \Gamma$

(*ii*) $d(u\alpha m\beta v + v\alpha m\beta u) = d(u)\alpha m\beta v + u\alpha d(m)\beta v + u\alpha m\beta d(v) + d(v)\alpha m\beta u + v\alpha d(m)\beta u + v\alpha m\beta d(u)$ for all $u, v \in U; m \in M$ and $\alpha, \beta \in \Gamma$.

Proof. By the definition of (*U*,*M*)-derivation of M , we have

$$d(u\alpha m + s\alpha u) = d(u)\alpha m + u\alpha d(m) + d(s)\alpha u + s\alpha d(u),\ for\, all\ u \in U; m, s \in M; \alpha \in \Gamma.$$

Replacing m and s by $(2u)\beta m + m\beta(2u)$ and let
$w = u\alpha((2u)\beta m + m\beta(2u)) + ((2u)\beta m + m\beta(2u))\alpha u$. Then using the definition of (U, M) -derivation and the condition (*), we get

$$\begin{aligned} d(w) &= 2(d(u)\alpha(u\beta m + m\beta u) + u\alpha d(u\beta m + m\beta u) + d(u\beta m + m\beta u)\alpha u + (u\beta m + m\beta u)\alpha d(u)) \\ &= 2(d(u)\alpha u\beta m + d(u)\alpha m\beta u + u\alpha d(u)\beta m + u\alpha u\beta d(m) + u\alpha d(m)\beta u + u\alpha m\beta d(u) \\ &+ d(u)\beta m\alpha u + u\beta d(m)\alpha u + d(m)\beta u\alpha u + m\beta d(u)\alpha u + u\beta m\alpha d(u) + m\beta u\alpha d(u)) \\ &= 2(d(u)\alpha u\beta m + d(u)\alpha m\beta u + u\alpha d(u)\beta m + u\alpha u\beta d(m) + u\alpha d(m)\beta u + u\alpha m\beta d(u) \\ &+ d(u)\alpha m\beta u + u\alpha d(m)\beta u + d(m)\alpha u\beta u + m\alpha d(u)\beta u + u\alpha m\beta d(u) + m\alpha u\beta d(u)). \end{aligned} \tag{1}$$

Also, we have

$$\begin{aligned} d(w) &= d((2u\alpha u)\beta m + m\beta(2u\alpha u)) + 2d(u\alpha m\beta u) + 2d(u\beta m\alpha u) \\ &= 2(d(u)\alpha u\beta m + u\alpha d(u)\beta m + u\alpha u\beta d(m) + d(m)\beta u\alpha u + m\beta d(u)\alpha u + m\beta u\alpha d(u) \\ &+ 2d(u\alpha m\beta u) + 2d(u\alpha m\beta u) \\ &= 2(d(u)\alpha u\beta m + u\alpha d(u)\beta m + u\alpha u\beta d(m) + d(m)\alpha u\beta u + m\alpha d(u)\beta u + m\alpha u\beta d(u)) \\ &+ 4d(u\alpha m\beta u). \end{aligned} \tag{2}$$

By comparing (1) and (2), and remembering that M is 2-torsion free, we obtain

$$d(u\alpha m\beta u) = d(u)\alpha m\beta u + u\alpha d(m)\beta u + u\alpha m\beta d(u), \forall\, u \in U; m \in M; \alpha, \beta \in \Gamma. \tag{3}$$

If we linearize (3) on u , then (ii) is obtained.

Definition 2. *For a* (*U*,*M*)-derivation d*, we define*
$\phi_\alpha(u,m) = d(u\alpha m) - d(u)\alpha m - u\alpha d(m)$ *for all* $u \in U, m \in M$ *and* $\alpha \in \Gamma$.

Lemma 2. *Let* d *be a* (*U*,*M*)-derivation *of a* Γ*-ring* M. *For all* $u, v \in U; m, n \in M$ *and* $\alpha \in \Gamma$, *the following statements are true:*
$(i)\ \phi_\alpha(m,u) = -\phi_\alpha(u,m); (ii)\ \phi_\alpha(u+v,m) = \phi_\alpha(u,m) + \phi_\alpha(v,m);$
$(iii)\ \phi_\alpha(u,m+n) = \phi_\alpha(u,m) + \phi_\alpha(u,n); (iv)\ \phi_{\alpha+\beta}(u,m) = \phi_\alpha(u,m) + \phi_\beta(u,m)$.

Proof. (i) Using Definition 2, we get

$$\begin{aligned}\phi_\alpha(u,m) + \phi_\alpha(m,u) &= d(u\alpha m) - d(u)\alpha m - u\alpha d(m) + d(m\alpha u) - d(m)\alpha a - m\alpha d(u)\\ &= d(u\alpha m + m\alpha u) - d(u)\alpha m - u\alpha d(m) - d(m)\alpha u - m\alpha d(u)\\ &= d(u)\alpha m + d(m)\alpha a + u\alpha d(m) + m\alpha d(u) - d(u)\alpha m - u\alpha d(m)\\ &\quad - d(m)\alpha u - m\alpha d(u) = 0.\end{aligned}$$

$\Rightarrow \phi_\alpha(m,u) = -\phi_\alpha(u,m).$

(ii) By the definition of (U,M)-derivation of M, we obtain

$$\begin{aligned}\phi_\alpha(u+v,m) &= d((u+v)\alpha m) - d(u+v)\alpha m - (u+v)\alpha d(m)\\ &= d(u\alpha m + v\alpha m) - d(u)\alpha m - d(v)\alpha m - u\alpha d(m) - v\alpha d(m)\\ &= d(u\alpha m) - d(u)\alpha m - u\alpha d(m) + d(v\alpha m) - d(v)\alpha m - v\alpha d(m)\\ &= \phi_\alpha(u,m) + \phi_\alpha(v,m).\end{aligned}$$

The proofs of (iii) and (iv) are straight forward and left to the reader.

Lemma 3. *Let* U *be a nonzero admissible Lie ideal of a 2-torsion free prime* Γ*-ring* M. *Then* U *contains a nonzero ideal of* M.

Proof. Since U is a noncentral Lie ideal of M, if $x, y \in U$ are any two elements, then $x\alpha y - y\alpha x \neq 0$ for every $\alpha \in \Gamma$. For any $m \in M$, using the condition (*) we get

$$\begin{aligned}x\alpha(y\beta m) - (y\beta m)\alpha x &= x\alpha(y\beta m) - y\alpha x\beta m + y\alpha x\beta m - (y\beta m)\alpha x\\ &= (x\alpha y - y\alpha x)\beta m + y\beta x\alpha m - y\beta m\alpha x\\ &= (x\alpha y - y\alpha x)\beta m + y\beta(x\alpha m - m\alpha x) \in U.\end{aligned}$$

Since U is a square closed Lie ideal of *M*, $2y\beta(x\alpha m - m\alpha x) \in U$. This leads us to

$2(x\alpha y - y\alpha x)\beta m \in U$ for all $m \in M$. Now for any $m, s \in M$, we have

$$(2(x\alpha y - y\alpha x)\beta m)\alpha s - s\alpha(2(x\alpha y - y\alpha x)\beta m) \in U; (2(x\alpha y - y\alpha x)\beta m)\alpha s \in U.$$

This implies,

$$s\alpha(2(x\alpha y - y\alpha x))\beta m \in U, \forall\, m, s \in M; \alpha, \beta \in \Gamma.$$

Let $I = M\Gamma 2(x\alpha y - y\alpha x)\Gamma M$. Then it is clear that I is an ideal contained in U. Now, we have to show that I is nonzero. Suppose that $I = 0$. By the 2-torsion freeness of M, $x\alpha y = y\alpha x$ which is a contradiction. Therefore, I is a nonzero ideal of M.

Lemma 4. *Let U be a Lie ideal of a prime Γ-ring M such that $U \not\subseteq Z(M)$. Then there exist elements $a, b \in U$ such that $[a,b]_\alpha = a\alpha b - b\alpha a \neq 0$.*

Proof. Assume that $[x, y]_\alpha = 0$ for every $x, y \in U$ and $\alpha \in \Gamma$. This gives $[U,U]_\Gamma = 0$, a contradiction to our assumption. So, there exist elements $a, b \in U$ such that $[a,b]_\alpha = a\alpha b - b\alpha a \neq 0$.

Lemma 5. *Assume that U is an admissible Lie ideal of a 2-torsion free prime Γ-ring M. If $t\alpha v\beta v + v\beta v\alpha t = 0$, for any $t \in M; v \in U$ and $\alpha, \beta \in \Gamma$, then $t = 0$.*

Proof. Since $t\alpha v\beta v + v\beta v\alpha t = 0$ for all $v \in U, t \in M$ and $\alpha, \beta \in \Gamma$. Linearizing on v, where $u \in U$, we get

$$\begin{aligned} 0 &= t\alpha(u+v)\beta(u+v) + (u+v)\beta(u+v)\alpha t \\ &= t\alpha(u\beta u + u\beta v + v\beta u + v\beta v) + (u\beta u + u\beta v + v\beta u + v\beta v)\alpha t \\ &= t\alpha(u\beta v + v\beta u) + (u\beta v + v\beta u)\alpha t. \end{aligned}$$

Replacing v by $v\alpha v$, we get

$$t\alpha(u\beta v\alpha v + v\alpha v\beta u) + (u\beta v\alpha v + v\alpha v\beta u)\alpha t = 0. \tag{4}$$

Applying $t\alpha v\beta v + v\beta v\alpha t = 0$ in (4), and using the condition (*) we get

$$t\alpha u\beta v\alpha v - v\alpha v\beta t\alpha u - u\alpha t\beta v\alpha v + v\alpha v\beta u\alpha t = 0.$$

$$\Rightarrow (t\alpha u - u\alpha t)\beta v\alpha v - v\alpha v\beta(t\alpha u - u\alpha t) = 0.$$

Therefore,

$$[t,u]_\alpha \beta v\alpha v - v\alpha v\beta[t,u]_\alpha = 0. \tag{5}$$

Again applying $t\alpha v\beta v + v\beta v\alpha t = 0$ in (5), we get

$$[t,u]_\alpha \beta v\alpha v-(-[t,u]_\alpha \beta v\alpha v)=0.$$
$$\Rightarrow 2[t,u]_\alpha \beta v\alpha v=0.$$

By the 2-torsion freeness of M,

$$[t,u]_\alpha \beta v\alpha v=0, \forall\, u,v\in U; t\in M; \alpha,\beta\in\Gamma.$$

This implies,

$$[M,U]_\Gamma \Gamma(v\alpha v)=0.$$

By Lemma 3, U contains a nonzero ideal I of M and this gives us, $[M,U]_\Gamma \Gamma I\Gamma(v\alpha v)=0$. Therefore, $[M,U]_\Gamma \Gamma M\Gamma I\Gamma(v\alpha v)\subseteq [M,U]_\Gamma \Gamma I\Gamma(v\alpha v)=0.$ Since M is prime, so $I\Gamma(v\alpha v)=0$ or $[M,U]_\Gamma=0$. If $I\Gamma(v\alpha v)=0$, then for $I\neq 0$ and by Lemma 4, we get $U=0$, which is a contradiction. Therefore, $[M,U]_\Gamma=0$, that is $t\beta v-v\beta t=0$ for all $v\in U, t\in M, \beta\in\Gamma$. Since $t\alpha v\beta v+v\beta v\alpha t=0,$ and applying $t\beta v=v\beta t,$ we get

$$0=t\alpha v\beta v+v\alpha v\beta t=t\alpha v\beta v+v\alpha t\beta v=t\alpha v\beta v+t\alpha v\beta v=2t\alpha v\beta v.$$

By the 2-torsion freeness of M, $t\alpha v\beta v=0$ for all $v\in U, t\in M, \beta\in\Gamma$. Linearizing $t\alpha v\beta v=0$ on v, where $u\in U,$ we get

$$0 = t\alpha(u+v)\beta(u+v)=t\alpha(u\beta v+v\beta u).$$

This implies,

$$t\alpha(u\beta v+v\beta u)\gamma u\alpha t=0.$$
$$\Rightarrow t\alpha u\beta v\gamma u\alpha t+t\alpha v\beta u\gamma u\alpha t=0.$$

Since $u\gamma u\alpha t=0$ and $t\alpha u=u\alpha t$, we get $(t\alpha u)\beta v\gamma(t\alpha u)=0.$

By the primeness of M, we get $t\alpha u=0$. Since $u\beta m-m\beta u\in U$ for all $u\in U, m\in M, \beta\in\Gamma,$ we gt $t\alpha(u\beta m-m\beta u)=0$, that is, $t\alpha u\beta m-t\alpha m\beta u=0.$ This implies, $t\alpha m\beta u=0$. But $u\neq 0$ and M is prime, consequently, $t=0$.

Lemma 6. *Let U be an admissible Lie ideal of a 2-torsion free prime Γ-ring M, and d be a* (*U*,*M*)-derivation *of M. Then $\phi_\beta(u\alpha u,m)=0$ for all $u\in U, m\in M$ and $\alpha,\beta\in\Gamma$.*

Proof. By Theorem 1, we have $\phi_\alpha(u,v)=0$ for all $u,v\in U;\alpha\in\Gamma$. Thus for all

$u \in U, m \in M$ and $\alpha, \beta \in \Gamma$, we obtain

$$\begin{aligned} 0 &= \phi_\alpha(u, u\beta m - m\beta u) \\ &= d(u\alpha(u\beta m - m\beta u)) - d(u)\alpha(u\beta m - m\beta u) - u\alpha d(u\beta m - m\beta u) \\ &= d(u\alpha u\beta m - u\alpha m\beta u) - d(u)\alpha(u\beta m - m\beta u) - u\alpha d(u\beta m - m\beta u) \end{aligned}$$

$$\begin{aligned} &= d(u\alpha u\beta m) - d(u\alpha m\beta u) - d(u)\alpha u\beta m + d(u)\alpha m\beta u - u\alpha(d(u)\beta m + u\beta d(m) - d(m)\beta u - m\beta d(u)) \\ &= d(u\alpha u\beta m) - d(u)\alpha m\beta u - u\alpha d(m)\beta u - u\alpha m\beta d(u) - d(u)\alpha u\beta m + d(u)\alpha m\beta u \\ &\quad - u\alpha d(u)\beta m - u\alpha u\beta d(m) + u\alpha d(m)\beta u + u\alpha m\beta d(u) \\ &= d(u\alpha u\beta m) - d(u)\alpha u\beta m - u\alpha d(u)\beta m - u\alpha u\beta d(m) \\ &= d((u\alpha u)\beta m) - d(u\alpha u)\beta m - -(u\alpha u)\beta d(m) = \phi_\beta(u\alpha u, m). \end{aligned}$$

Now we state and prove our main results as theorem.

Theorem 1. *Let U be an admissible Lie ideal of a 2-torsion free prime Γ-ring M, and let d be a* (*U,M*)-derivation *of M. Then $\phi_\alpha(u,v) = 0$ for all $u, v \in U$ and $\alpha \in \Gamma$.*

Proof. Let $x = 4(u\alpha v\beta[u,v]_\alpha\gamma v\alpha u + v\alpha u\beta[u,v]_\alpha\gamma u\alpha v)$. Then using Lemma 1(ii), we get

$$\begin{aligned} d(x) &= d((2u\alpha v)\beta[u,v]_\alpha\gamma(2v\alpha u) + (2v\alpha u)\beta[u,v]_\alpha\gamma(2u\alpha v)) \\ &= d(2u\alpha v)\beta[u,v]_\alpha\gamma(2v\alpha u) + 2u\alpha v d(\beta[u,v]_\alpha)\gamma 2v\alpha u + 2u\alpha v\beta[u,v]_\alpha\gamma d(2v\alpha u) \\ &\quad + d(2v\alpha u)\beta[u,v]_\alpha\gamma(2u\alpha v) + 2v\alpha u d(\beta[u,v]_\alpha)\gamma 2u\alpha v + 2v\alpha u\beta[u,v]_\alpha\gamma d(2u\alpha v). \end{aligned}$$

On the other hand, using Lemma 1(i), we get

$$\begin{aligned} d(x) &= d(u\alpha(4v\beta[u,v]_\alpha\gamma v)\alpha u + v\alpha(4u\beta[u,v]_\alpha\gamma u)\alpha v) \\ &= d(u)\alpha 4v\beta[u,v]_\alpha\gamma v\alpha u + u\alpha d(4v\beta[u,v]_\alpha\gamma v)\alpha u + u\alpha 4v\beta[u,v]_\alpha\gamma v\alpha d(u) \\ &\quad + d(v)\alpha 4u\beta[u,v]_\alpha\gamma u\alpha v + v\alpha d(4u\beta[u,v]_\alpha\gamma u)\alpha v + v\alpha 4u\beta[u,v]_\alpha\gamma u\alpha d(v) \\ &= 4d(u)\alpha v\beta[u,v]_\alpha\gamma v\alpha u + 4u\alpha d(v)\beta[u,v]_\alpha\gamma v\alpha u + 4u\alpha v d(\beta[u,v]_\alpha)\gamma v\alpha u \\ &\quad + 4u\alpha v\beta[u,v]_\alpha\gamma d(v)\alpha u + 4u\alpha v\beta[u,v]_\alpha\gamma v\alpha d(u) + 4d(v)\alpha u\beta[u,v]_\alpha\gamma u\alpha v \\ &\quad + 4v\alpha d(u)\beta[u,v]_\alpha\gamma u\alpha v + 4v\alpha u d(\beta[u,v]_\alpha)\gamma u\alpha v + 4v\alpha u\beta[u,v]_\alpha\gamma d(u)\alpha v \\ &\quad + 4v\alpha u\beta[u,v]_\alpha\gamma u\alpha d(v). \end{aligned}$$

Equating these two expressions for $d(x)$ and using Definition 2, we obtain

$$4(d(u\alpha v)-d(u)\alpha v-u\alpha d(v))\beta[u,v]_\alpha\gamma v\alpha u+4(d(v\alpha u)-d(v)\alpha u-v\alpha d(u))\beta[u,v]_\alpha\gamma u\alpha v$$
$$+4u\alpha v\beta[u,v]_\alpha\gamma(d(v\alpha u)-d(v)\alpha u-v\alpha d(u))+4v\alpha u\beta[u,v]_\alpha\gamma(d(u\alpha v)-d(u)\alpha v-u\alpha d(v))=0.$$

$$\Rightarrow 4(\phi_\alpha(u,v)\beta[u,v]_\alpha\gamma v\alpha u+\phi_\alpha(v,u)\beta[u,v]_\alpha\gamma u\alpha v+u\alpha v\beta[u,v]_\alpha\gamma\phi_\alpha(v,u)$$
$$+v\alpha u\beta[u,v]_\alpha\gamma\phi_\alpha(u,v))=0.$$

Using Lemma 2(i), we get

$$4(\phi_\alpha(u,v)\beta[u,v]_\alpha\gamma v\alpha u-\phi_\alpha(u,v)\beta[u,v]_\alpha\gamma u\alpha v-u\alpha v\beta[u,v]_\alpha\gamma\phi_\alpha(u,v)+v\alpha u\beta[u,v]_\alpha\gamma\phi_\alpha(u,v))=0.$$
$$\Rightarrow 4(\phi_\alpha(u,v)\beta[u,v]_\alpha\gamma[u,v]_\alpha+[u,v]_\alpha\beta[u,v]_\alpha\gamma\phi_\alpha(u,v))=0.$$

Using the condition (*) and 2-torsion freeness of M,

$$\phi_\alpha(u,v)\gamma[u,v]_\alpha\beta[u,v]_\alpha+[u,v]_\alpha\beta[u,v]_\alpha\gamma\phi_\alpha(u,v)=0,\ \textit{for all } u,v\in U,\alpha,\beta,\gamma\in\Gamma.$$

Since $U\not\subseteq Z(M)$, therefore, $[u,v]_\alpha\neq 0$ for all $u,v\in U$ and $\alpha\in\Gamma$. Hence by Lemma 5, we obtain $\phi_\alpha(u,v)=0$ for all $u,v\in U$ and $\alpha\in\Gamma$.

Theorem 2. *Let U be a square closed Lie ideal of a 2-torsion free prime Γ-ring M, and d be a (U,M)-*derivation *of M. Then $d(u\alpha m)=d(u)\alpha m+u\alpha d(m)$ for all $u\in U,m\in M$ and $\alpha\in\Gamma$.*

Proof. Since d is a (*U*,*M*)-derivation of a prime Γ-ring M, so for all $u\in U,m\in M$ and $\alpha,\beta\in\Gamma$, we have

$$d(u\alpha(u\beta m)+(u\beta m)\alpha u)=d(u)\alpha u\beta m+u\alpha d(u\beta m)+d(u\beta m)\alpha u+u\beta m\alpha d(u).\quad (6)$$

On the other hand

$$d(u\alpha u\beta m+u\beta m\alpha u)=d(u\alpha u\beta m)+d(u)\beta m\alpha u+u\beta d(m)\alpha u+u\beta m\alpha d(u).\quad (7)$$

From Lemma 6, we have

$$\phi_\beta(u\alpha u,m)=0,\forall\ u\in U;m\in M;\alpha,\beta\in\Gamma.$$
$$\Rightarrow d(u\alpha u\beta m)-d(u)\alpha u\beta m-u\alpha d(u)\beta m-u\alpha u\beta d(m)=0.$$
$$\Rightarrow d(u\alpha u\beta m)=d(u)\alpha u\beta m+u\alpha d(u)\beta m+u\alpha u\beta d(m).\quad (8)$$

Now, using (8) in (7), we get

$$d(u\alpha u\beta m+u\beta m\alpha u)=d(u)\alpha u\beta m+u\alpha d(u)\beta m+u\alpha u\beta d(m)+d(u)\beta m\alpha u$$
$$+u\beta d(m)\alpha u+u\beta m\alpha d(u). \quad (9)$$

Comparing (6) and (9), we get

$$u\alpha d(u\beta m)+d(u\beta m)\alpha u=u\alpha d(u)\beta m+u\alpha u\beta d(m)+d(u)\beta m\alpha u+u\beta d(m)\alpha u.$$

Using Definition 2, we obtain

$$u\alpha\phi_\beta(u,m)+\phi_\beta(u,m)\alpha u=0, \quad \textit{for all } u\in U, m\in M;\alpha,\beta\in\Gamma. \quad (10)$$

Linearizing (10) on u, and using (10)

$$(u+v)\alpha\phi_\beta(u+v,m)+\phi_\beta(u+v,m)\alpha(u+v)=0.$$
$$\Rightarrow u\alpha\phi_\beta(u,m)+u\alpha\phi_\beta(v,m)+v\alpha\phi_\beta(u,m)+v\alpha\phi_\beta(v,m)$$
$$+\phi_\beta(u,m)\alpha u+\phi_\beta(u,m)\alpha v+\phi_\beta(v,m)\alpha u+\phi_\beta(v,m)\alpha v=0.$$
$$\Rightarrow u\alpha\phi_\beta(v,m)+v\alpha\phi_\beta(u,m)+\phi_\beta(u,m)\alpha v+\phi_\beta(v,m)\alpha u=0. \quad (11)$$

Replacing v by $v\gamma v$ in (11) and using Lemma 6, we get

$$(v\gamma v)\alpha\phi_\beta(u,m)+\phi_\beta(u,m)\alpha(v\gamma v)=0.$$

If $U\not\subseteq Z(M)$, using Lemma 5, $\phi_\beta(u,m)=0$ for all $u\in U, m\in M$ and $\beta\in\Gamma$. If $U\subseteq Z(M)$, by the 2-torsion freeness of M, $(v\gamma v)\alpha\phi_\beta(u,m)=0.$ Therefore, $0=c\delta(v\gamma v)\alpha\phi_\beta(u,m)=(v\gamma v)\delta c\alpha\phi_\beta(u,m),$ where $c\in M$ and $\delta\in\Gamma$. As M is prime, so $v\gamma v=0$ or $\phi_\beta(u,m)=0$. But $v\neq 0$, hence $\phi_\beta(u,m)=0$ for all $u\in U, m\in M$ and $\beta\in\Gamma$. This completes the proof of the theorem.

Corollary 1. *Let M be a 2-torsion free prime Γ-ring satisfying the condition (*), and U be a square closed Lie ideal of M. Then every Jordan derivation d on U of M is a derivation on U of M.*

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