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DERIVATIONS ON LIE IDEALS OF COMPLETELY SEMIPRIME Γ-RINGS

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Abstract

The aim of the paper is to prove the following theorem concerning a class of Γ-rings. Let M be a 2-torsion free completely semiprime Γ-ring satisfying the condition $a\alpha b\beta c = a\beta b\alpha c$, for all $a,b,c \in M$ and $\alpha, \beta \in \Gamma$; U be an admissible Lie ideal of M. If $d: M \to M$ is a Jordan derivation on U of M, then d is a derivation on U of M.

Key words: Jordan derivation, admissible Lie ideal, completely semiprime Γ-ring.

Introduction

Let M and Γ be additive abelian groups. If there is a mapping $M \times \Gamma \times M \to M$ such that the conditions

- $(x+y)\alpha z = x\alpha z + y\alpha z, x(\alpha+\beta)y = x\alpha y + x\beta y, x\alpha(y+z) = x\alpha y + x\alpha z$
- $(x\alpha y)\beta z = x\alpha(y\beta z)$

are satisfied for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$, then M is called a Γ-ring in the sense of Barnes (1966). The concept of a Γ-ring was first introduced by Nobusawa (1964) and afterwards it was generalized by Barnes (1966). This concept is more general than that of a ring. A Γ-ring M is called semiprime if $a\Gamma M\Gamma a = 0$ (with $a \in M$) implies $a = 0$ and M is called completely semiprime if $a\Gamma a = 0$ (with $a \in M$) implies $a = 0$. A Γ-ring M is 2-torsion free if $2a = 0$ implies $a = 0, \forall a \in M$. For any $x, y \in M$ and $\alpha \in \Gamma$, we denote the commutator $x\alpha y - y\alpha x$ by $[x, y]_\alpha$. An additive subgroup $U \subseteq M$ is said to be a Lie ideal of M if whenever $u \in U, m \in M$ and $\alpha \in \Gamma$, then $[u, m]_\alpha \in U$. In the main results of this article we assume that the Lie ideal U satisfies $u\alpha u \in U, \forall u \in U, \alpha \in \Gamma$. A Lie ideal of this type is called a square closed Lie ideal. Furthermore, if the Lie ideal U is square closed and $U \not\subseteq Z(M)$, where $Z(M)$ denotes the centre of M then U is called an admissible Lie ideal of M.

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Let M be a Γ-ring. An additive mapping $d: M \to M$ is a derivation if $d(a\alpha b) = d(a)\alpha b + a\alpha d(b)$ and a Jordan derivation if if

$d(a\alpha a) = d(a)\alpha a + a\alpha d(a), \forall a, b \in M$ and $\alpha \in \Gamma$. Throughout the article, we use the condition $a\alpha b\beta c = a\beta b\alpha c, \forall a, b, c \in M$ and $\alpha, \beta \in \Gamma$ and this is denoted by (*). The relationship between usual derivations and Lie ideals of prime rings have been extensively studied in the last 40 years. In particular, when this relationship involves the action of the derivations on Lie ideals.

Awtar (1984) extended a well known result proved by Herstein (1957) to Lie ideals which states that "every Jordan derivation on a 2-torsion free prime ring is a derivation". In fact, Awtar (1984) proved that if $U \not\subseteq Z(R)$ is a square closed Lie ideal of a 2-torsion free prime ring R and $d: R \to R$ is an additive mapping such that $d(u^2) = d(u)u + ud(u), \forall u \in U$ then $d(uv) = d(u)v + ud(v), \forall u, v \in U$. Ashraf and Rehman (2000) studied on Lie ideals and Jordan left derivations of prime rings. They proved that if $d: R \to R$ is an additive mapping on a 2-torsion free prime ring R satisfying $d(u^2) = 2ud(u), \forall u \in U$, where U is a Lie ideal of R such that $u^2 \in U, \forall u \in U$, then $d(uv) = d(u)v + ud(v), \forall u, v \in U$. Halder and Paul (2012) extended the results of Ceven (2002) in Lie ideals. We have generalized the Awtar (1984) result in completely semiprime Γ-rings.

Jordan Derivations on Lie Ideals of Completely Semiprime Γ-Rings

Definition 1. Let M be a Γ-ring and U be a Lie ideal of M. An additive mapping $d: M \to M$ is said to be a Jordan derivation on a Lie ideal U of M if $d(u\alpha u) = d(u)\alpha u + u\alpha d(u), \forall u \in U$ and $\alpha \in \Gamma$

Example 1. Let R be a ring of characteristic 2 having a unity element 1.

Let $M = M_{1,2}(R)$ and $\Gamma = \left\{ \begin{pmatrix} n.1 \\ n.1 \end{pmatrix} : n \in Z \right\}$, where n is not divisible by 2 .

Then M is a Γ-ring. Let $N = \{(x, x) : x \in R\} \subseteq M$.

Then $\forall (x, x) \in N, (a, b) \in M$ and $\begin{pmatrix} n \\ n \end{pmatrix} \in \Gamma$,

$$(x, x)\begin{pmatrix} n \\ n \end{pmatrix}(a, b) - (a, b)\begin{pmatrix} n \\ n \end{pmatrix}(x, x) = (xna - bnx, xnb - anx)$$

$$= (xna - 2bnx + bnx, bnx - 2anx + xna)$$

$$= (xna + bnx, bnx + xna) \in N.$$

Therefore, N is a Lie ideal of *M*.

Example 2. Let M be a Γ-ring satisfying the condition (*) and let U be a Lie ideal of M. Let $a \in M$ and $\alpha \in \Gamma$ be fixed elements. Define $d : M \to M$ by $d(x) = a\alpha x - x\alpha a, \forall x \in U$. Then $\forall y \in U$ and $\beta \in \Gamma$, we have

$$\begin{aligned} d(x\beta y) &= a\alpha x\beta y - x\beta y\alpha a \\ &= a\alpha x\beta y - x\alpha a\beta y + x\alpha a\beta y - x\beta y\alpha a \\ &= (a\alpha x - x\alpha a)\beta y + x\beta a\alpha y - x\beta y\alpha a \\ &= (a\alpha x - x\alpha a)\beta y + x\beta(a\alpha y - y\alpha a) \\ &= d(x)\beta y + x\beta d(y), \end{aligned}$$

Therefore, $d : M \to M$ is a derivation on U.

Example 3. Let M be a Γ-ring and U be a Lie ideal of M. Let $d : M \to M$ be a derivation on U. Suppose $M_1 = \{(x,x) : x \in M\}$ and $\Gamma_1 = \{(\alpha,\alpha) : \alpha \in \Gamma\}$. Define addition and multiplication on M_1 by $(x,x) + (y,y) = (x+y, x+y)$ and $(x,x)(\alpha,\alpha)(y,y) = (x\alpha y, x\alpha y)$. Then M_1 is a Γ_1-ring. Suppose $U_1 = \{(u,u) : u \in U\}$. Then

$$(u,u)(\alpha,\alpha)(x,x) - (x,x)(\alpha,\alpha)(u,u) = (u\alpha x, u\alpha x) - (x\alpha u, x\alpha u)$$
$$= (u\alpha x - x\alpha u, u\alpha x - x\alpha u) \in U_1$$

for $u\alpha x - x\alpha u \in U$. Hence U_1 is a Lie ideal of M_1. If we define a mapping $D : M_1 \to M_1$ on U_1 by $D((u,u)) = (d(u), d(u))$. Then it is clear that D is a Jordan derivation on U_1 which is not a derivation on U_1.

Lemma 1. Let M be a Γ-ring and U be a Lie ideal of M such that $u\alpha u \in U, \forall u \in U$ and $\alpha \in \Gamma$. If d is a Jordan derivation on U of M, then $\forall a,b,c \in U$ and $\alpha, \beta \in \Gamma$, the following statements hold:

(i) $d(a\alpha b + b\alpha a) = d(a)\alpha b + d(b)\alpha a + a\alpha d(b) + b\alpha d(a)$.

(ii) $d(a\alpha b\beta a + a\beta b\alpha a) = d(a)\alpha b\beta a + d(a)\beta b\alpha a + a\alpha d(b)\beta a + a\beta d(b)\alpha a$
$+ a\alpha b\beta d(a) + a\beta b\beta d(a)$

In particular, if M is 2-torsion free and satisfies the condition (*), then

(iii) $d(a\alpha b\beta a) = d(a)\alpha b\beta a + a\alpha d(b)\beta a + a\alpha b\beta d(a)$.

(iv) $d(a\alpha b\beta c + c\alpha b\beta a) = d(a)\alpha b\beta c + d(c)\alpha b\beta a + a\alpha d(b)\beta c + c\alpha d(b)\beta a$
$+ a\alpha b\beta d(c) + c\alpha b\beta d(a)$.

Proof. Since U is a Lie ideal satisfying the condition $a\alpha a \in U, \forall a \in U, \alpha \in \Gamma$. For $a, b \in U, \alpha \in \Gamma, (a\alpha b + b\alpha a) = (a+b)\alpha(a+b) - (a\alpha a + b\alpha b)$ and so $(a\alpha b + b\alpha a) \in U$.

Also, $[a,b]_\alpha = a\alpha b - b\alpha a \in U$ and it follows that $2a\alpha b \in U$.

Hence $4a\alpha b\beta c = 2(2a\alpha b)\beta c \in U, \forall a,b,c \in U, \alpha, \beta \in \Gamma$. Thus,

$$\begin{aligned} d(a\alpha b + b\alpha a) &= d((a+b)\alpha(a+b) - (a\alpha a + b\alpha b)) \\ &= d(a+b)\alpha(a+b) + (a+b)\alpha d(a+b) - d(a)\alpha a \\ &\quad - a\alpha d(a) - d(b)\alpha b - b\alpha d(b) \\ &= d(a)\alpha a + d(a)\alpha b + d(b)\alpha a + d(b)\alpha b + a\alpha d(a) + a\alpha d(b) \\ &\quad + b\alpha d(a) + b\alpha d(b) - d(a)\alpha a - a\alpha d(a) - d(b)\alpha b - b\alpha d(b) \\ &= d(a)\alpha b + a\alpha d(b) + d(b)\alpha a + b\alpha d(a). \end{aligned}$$

Replacing $a\beta b + b\beta a$ for b in (i), we get

$$\begin{aligned} &d(a\alpha(a\beta b + b\beta a) + (a\beta b + b\beta a)\alpha a) = d(a)\alpha(a\beta b + b\beta a) + a\alpha d(a\beta b + b\beta a) \\ &+ d(a\beta b + b\beta a)\alpha a + (a\beta b + b\beta a)\alpha d(a). \end{aligned}$$

This implies,

$$\begin{aligned} &d(a\alpha a)\beta b + (a\alpha a)\beta d(b) + d(b)\beta(a\alpha a) + b\beta d(a\alpha a) + d(a\alpha b\beta a + a\beta b\alpha a) \\ &= d(a)\alpha a\beta b + d(a)\alpha b\beta a + a\alpha d(a)\beta b + a\alpha a\beta d(b) + a\alpha d(b)\beta a + a\alpha b\beta d(a) \\ &+ d(a)\beta b\alpha a + a\beta d(b)\alpha a + d(b)\beta a\alpha a + b\beta d(a)\alpha a + a\beta b\alpha d(a) + b\beta a\alpha d(a). \end{aligned}$$

This implies,

$$\begin{aligned} &d(a)\alpha a\beta b + a\alpha d(a)\beta b + a\alpha a\beta d(b) + d(b)\beta a\alpha a + b\beta d(a)\alpha a + b\beta a\alpha d(a) \\ &+ d(a\alpha b\beta a + a\beta b\alpha a) \\ &= d(a)\alpha a\beta b + d(a)\alpha b\beta a + a\alpha d(a)\beta b + a\alpha a\beta d(b) + a\alpha d(b)\beta a + a\alpha b\beta d(a) \\ &+ d(a)\beta b\alpha a + a\beta d(b)\alpha a + d(b)\beta a\alpha a + b\beta d(a)\alpha a + a\beta b\alpha d(a) + b\beta a\alpha d(a). \end{aligned}$$

Now, cancelling the like terms from both sides we get the required result. Using the condition (*) and since M is 2-torsion free, (iii) follows from (ii). And finally (iv) is obtained by replacing $a+c$ for a in (iii).

Definition 2. Let M be a 2-torsion free completely semiprime Γ-ring satisfying the condition (*) and U be a Lie ideal of M. Let d be a Jordan derivation on U of M. Then $\forall a,b \in U$ and $\alpha \in \Gamma$, we define $G_\alpha(a,b) = d(a\alpha b) - d(a)\alpha b - a\alpha d(b)$.

Lemma 2. Let M be 2-torsion free completely semiprime Γ-ring satisfying the condition (*) and U be a Lie ideal of M. Let d be a Jordan derivation on U of M. Then $\forall a,b,c \in U$ and $\alpha, \beta \in \Gamma$, the following statements hold.

(i) $G_\alpha(a,b) + G_\alpha(b,a) = 0$; (ii) $G_\alpha(a+b,c) = G_\alpha(a,c) + G_\alpha(b,c)$;

(iii) $G_\alpha(a,b+c) = G_\alpha(a,b) + G_\alpha(a,c)$; (iv) $G_{\alpha+\beta}(a,b) = G_\alpha(a,b) + G_\beta(a,b)$.

Remark 1. d is a derivation on U of M if and only if $G_\alpha(a,b) = 0, \forall a,b \in U$ and $\alpha \in \Gamma$.

Lemma 3. Let M be a 2-torsion free completely semiprime Γ-ring satisfying the condition (*) and U be a Lie ideal of M. If $u \in U$ such that $[u,[u,x]_\alpha]_\alpha = 0, \forall x \in M$ and $\alpha \in \Gamma$, then $[u,x]_\alpha = 0$.

Proof. We have $[u,[u,x]_\alpha]_\alpha = 0, \forall x \in M$ and $\alpha \in \Gamma$. For every $\beta \in \Gamma$, replacing x by $x\beta x$, we obtain

$$\begin{aligned} 0 &= [u,[u,x\beta x]_\alpha]_\alpha \\ &= [u, x\beta[u,x]_\alpha + [u,x]_\alpha \beta x]_\alpha \\ &= [u, x\beta[u,x]_\alpha]_\alpha + [u,[u,x]_\alpha \beta x]_\alpha \\ &= x\beta[u,[u,x]_\alpha]_\alpha + [u,x]_\alpha \beta[u,x]_\alpha + [u,[u,x]_\alpha]_\alpha \beta x + [u,x]_\alpha \beta[u,x]_\alpha \\ &= 2[u,x]_\alpha \beta[u,x]_\alpha. \end{aligned}$$

By the 2-torsion freeness of M, we obtain $[u,x]_\alpha \beta[u,x]_\alpha = 0$. Since M is completely semiprime Γ-ring, hence $[u,x]_\alpha = 0, \forall x \in M$ and $\alpha \in \Gamma$.

Lemma 4. Let M be a 2-torsion free completely semiprime Γ-ring satisfying the condition (*) and U be a commutative Lie ideal of M, then $U \subseteq Z(M)$.

Proof. Since U is a commutative Lie ideal of M, so we have $[u,[u,x]_\alpha]_\alpha = 0, \forall u \in U, x \in M$ and $\alpha \in \Gamma$. Then by Lemma 3, we get $[u,x]_\alpha = 0$. This implies that $U \subseteq Z(M)$.

Lemma 5. Let M be a 2-torsion free completely semiprime Γ-ring satisfying the condition (*). If $U \neq 0$ is a sub-Γ-ring and a Lie ideal of M, then either $U \subseteq Z(M)$ or U contains a non-zero ideal of M.

Proof. If U is commutative, then by Lemma 4, $U \subseteq Z(M)$. So let U be non-commutative, then for some $u, v \in M$ and $\alpha \in \Gamma$, we have $[u,v]_\alpha \in U$. Hence there exists an ideal J of M generated by $[u,v]_\alpha \neq 0$ and $J \subseteq U$.

Lemma 6. Let M be a 2-torsion free completely semiprime Γ-ring satisfying the condition (*). If $U \not\subseteq Z(M)$, then $Z(U) = Z(M)$.

Proof. $Z(U)$ is both a sub-Γ-ring and a Lie ideal of M such that $Z(U)$ does not contain non-zero ideal of M. Therefore in view of Lemma 5, we obtain that $Z(U) \subseteq Z(M)$. Hence $Z(U) = Z(M)$.

Lemma 7. Let M be a 2-torsion free completely semiprime Γ-ring satisfying the condition (*) and U be a Lie ideal of M, then $Z([U,U]_\Gamma) = Z(U)$.

Proof. Let $a \in M$ be any element. If $[a,[U,U]_\Gamma]_\Gamma = 0$, then we prove that . This yields that $Z([U,U]_\Gamma) = Z(U)$. If $[U,U]_\Gamma \not\subset Z(M)$, then by Lemma 6, $a \in Z(U)$. So a centralizes U. On the other hand, let $[U,U]_\Gamma \subseteq Z(M)$. Then we have $[u,[u,a]_\alpha]_\alpha = 0, \forall u \in U, a \in M$ and $\alpha \in \Gamma$. Thus, in view of Lemma 4, we obtain that $[u,a]_\alpha = 0, \forall u \in U, a \in M$ and $\alpha \in \Gamma$. This gives that $a \in Z(U)$. Hence we have the required result.

Lemma 8. Let M be 2-torsion free completely semiprime Γ-ring satisfying the condition (*) and U be a Lie ideal of M. If d is a Jordan derivation on U of M. Then $G_\alpha(a,b)\beta[a,b]_\alpha + [a,b]_\alpha \beta G_\alpha(a,b) = 0, \forall a,b \in U$ and $\alpha, \beta \in \Gamma$.

Proof. For any $a,b \in U$ and $\alpha, \beta \in \Gamma$, let $w = 4(a\alpha b\beta b\alpha a + b\alpha a\beta a\alpha b)$.

Then using Lemma 1(i)

$$\begin{aligned} d(w) &= d((2a\alpha b)\beta(2b\alpha a) + (2b\alpha a)\beta(2a\alpha b)) \\ &= 4d(a\alpha b)\beta(b\alpha a) + 4(a\alpha b)\beta d(b\alpha a) + 4d(b\alpha a)\beta(a\alpha b) \\ &\quad + 4(b\alpha a)\beta d(a\alpha b). \end{aligned}$$

On the other hand, using Lemma 1(iii)

$$\begin{aligned} d(w) &= d(2(a\alpha(2b\beta b)\alpha a)+2(b\alpha(2a\beta a)\alpha b)) \\ &= 2d(a)\alpha(2b\beta b)\alpha a+2a\alpha d(2b\beta b)\alpha a+2a\alpha(2b\beta b)\alpha d(a)+2d(b)\alpha(2a\beta a)\alpha b \\ &\quad +2b\alpha d(2a\beta a)\alpha b+2b\alpha(2a\beta a)\alpha d(b) \\ &= 4d(a)\alpha b\beta b\alpha a+4a\alpha d(b)\beta b\alpha a+4a\alpha b\beta d(b)\alpha a+4a\alpha b\beta b\alpha d(a) \\ &\quad +4d(b)\alpha a\beta a\alpha b+4b\alpha d(a)\beta a\alpha b+4b\alpha a\beta d(a)\alpha b+4b\alpha a\beta a\alpha d(b). \end{aligned}$$

Equating the two expressions for $d(w)$, we get

$$4(d(a\alpha b)-d(a)\alpha b-a\alpha d(b))\beta b\alpha a+4(d(b\alpha a)-d(b)\alpha a- b\alpha d(a))\beta a\alpha b+4a\alpha b\beta(d(b\alpha a)-d(b)\alpha a-b\alpha d(a)) + b\alpha a\beta(d(a\alpha b)-d(a)\alpha b-a\alpha d(b))=0.$$

Now using the Definition 2, we obtain

$$4G_\alpha(a,b)\beta b\alpha a+4G_\alpha(b,a)\beta a\alpha b+4a\alpha b\beta G_\alpha(b,a)+4b\alpha a\beta G_\alpha(a,b)=0.$$

Using Lemma 2(i), we have

$$4G_\alpha(a,b)\beta b\alpha a-4G_\alpha(a,b)\beta a\alpha b-4a\alpha b\beta G_\alpha(a,b)+4b\alpha a\beta G_\alpha(a,b)=0.$$

By the 2-torsion freeness of M, we get

$G_\alpha(a,b)\beta[a,b]_\alpha+[a,b]_\alpha\beta G_\alpha(a,b)=0, \forall a,b\in U$ and $\alpha,\beta\in\Gamma$.

Lemma 9. Let M be a 2-torsion free completely semiprime Γ-ring, U be a Lie ideal of M and let $a,b\in U$ and $\alpha\in\Gamma$. If $a\alpha b+b\alpha a=0$ then $a\alpha b=0=b\alpha a$.

Proof. Let $\delta\in\Gamma$ be any element.

Suppose that $a,b\in U$ and $\alpha\in\Gamma$ such that $a\alpha b+b\alpha a=0$

Using the relation $a\alpha b=-b\alpha a$ repeatedly, we get

$$\begin{aligned} 4(a\alpha b)\delta(a\alpha b) &= -4(b\alpha a)\delta(a\alpha b)=-4(b(\alpha a\delta)a)\alpha b \\ &= 4(a(\alpha a\delta)b)\alpha b=2a\alpha(2a\delta b)\alpha b \\ &= -2a\alpha(2b\delta a)\alpha b=-4(a\alpha b)\delta(a\alpha b). \end{aligned}$$

This implies,

$$8((a\alpha b)\delta(a\alpha b))=0.$$

Since M is 2-torsion free,

$$(a\alpha b)\delta(a\alpha b)=0.$$

Therefore,

$$(a\alpha b)\Gamma(a\alpha b) = 0.$$

By the complete semiprimeness of M, we get $a\alpha b = 0.$ Similarly, it can be shown that, $b\alpha a = 0.$

Corollary 1. Let M be a 2-torsion free completely semiprime Γ-ring satisfying the condition (*), U be a Lie ideal of M and d be a Jordan derivation on U of M. Then $\forall a,b \in U$ and $\alpha,\beta \in \Gamma : (i) G_\alpha(a,b)\beta[a,b]_\alpha = 0; (ii)[a,b]_\alpha \beta G_\alpha(a,b) = 0.$

Proof. Applying the result of Lemma 9 in that of Lemma 8, we obtain these results.

Lemma 10. Let M be a 2-torsion free completely semiprime Γ-ring satisfying the condition (*), U be a Lie ideal of M and d be a Jordan derivation on U of M. Then $\forall a,b,x,y \in U$ and $\alpha,\beta,\gamma \in \Gamma$:

$$(i) G_\alpha(a,b)\beta[x,y]_\alpha = 0; (ii)[x,y]_\alpha \beta G_\alpha(a,b) = 0$$
$$(iii) G_\alpha(a,b)\beta[x,y]_\gamma = 0; (iv)[x,y]_\gamma \beta G_\alpha(a,b) = 0.$$

Proof. (i) If we substitute for a in the Corollary 2.1 (i), then we get

$$G_\alpha(a+x,b)\beta[a+x,b]_\alpha = 0$$

Using Lemma 2(ii), we have

$$G_\alpha(a,b)\beta[a,b]_\alpha + G_\alpha(a,b)\beta[x,b]_\alpha + G_\alpha(x,b)\beta[a,b]_\alpha + G_\alpha(x,b)\beta[x,b]_\alpha = 0.$$

Now, using Corollary 1(i), we obtain

$$G_\alpha(a,b)\beta[x,b]_\alpha + G_\alpha(x,b)\beta[a,b]_\alpha = 0.$$

That is, $G_\alpha(a,b)\beta[x,b]_\alpha = -G_\alpha(x,b)\beta[a,b]_\alpha$.

Now,

$$(G_\alpha(a,b)\beta[x,b]_\alpha)\beta(G_\alpha(a,b)\beta[x,b]_\alpha) = -G_\alpha(a,b)\beta[x,b]_\alpha \beta G_\alpha(x,b)\beta[a,b]_\alpha = 0.$$

Hence, by the complete semiprimeness of M, we obtain

$$G_\alpha(a,b)\beta[x,b]_\alpha = 0.$$

Similarly, by replacing $b+y$ for b in this result, we get

$$G_\alpha(a,b)\beta[x,y]_\alpha = 0.$$

(ii) Proceeding in the same way as described above by the similar replacements successively in Corollary 1(ii), we obtain

$$[x,y]_\alpha \beta G_\alpha(a,b) = 0, \forall a,b,x,y \in U \text{ and } \alpha,\beta \in \Gamma.$$

(iii) Replacing $\alpha+\gamma$ for α in (i), we get

$$G_{\alpha+\gamma}(a,b)\beta[x,y]_{\alpha+\gamma} = 0.$$

By using Lemma 2(iv), we have

$$(G_\alpha(a,b)+G_\gamma(a,b))\beta([x,y]_\alpha+[x,y]_\gamma) = 0.$$

This implies,

$$G_\alpha(a,b)\beta[x,y]_\alpha + G_\alpha(a,b)\beta[x,y]_\gamma + G_\gamma(a,b)\beta[x,y]_\alpha + G_\gamma(a,b)\beta[x,y]_\gamma = 0.$$

Thus, using (i), we get

$$G_\alpha(a,b)\beta[x,y]_\gamma + G_\gamma(a,b)\beta[x,y]_\alpha = 0.$$

That is, $G_\alpha(a,b)\beta[x,y]_\gamma = -G_\gamma(a,b)\beta[x,y]_\alpha$.

Thus , we have

$$(G_\alpha(a,b)\beta[x,y]_\gamma)\beta(G_\alpha(a,b)\beta[x,y]_\gamma) = -G_\alpha(a,b)\beta[x,y]_\gamma\beta G_\gamma(a,b)\beta[x,y]_\alpha = 0.$$

Hence, by the complete semiprimeness of M , we obtain

$$G_\alpha(a,b)\beta[x,y]_\gamma = 0.$$

(iv) By performing the similar replacement in (ii)(as in the proof of (iii)), we get this result.

Now, we are ready to prove our main result.

Remark 2. If U is a commutative Lie ideal of M , then $U \subseteq Z(M)$. So by Lemma 2.1(i) and using 2-torsion freeness of M , we get $d(a\alpha b) = d(a)\alpha b + a\alpha d(b), \forall a,b \in U$ and $\alpha \in \Gamma$. Thus for the next results, we assume that $U \not\subseteq Z(M)$.

Theorem 1. Let M be a 2-torsion free completely semiprime Γ-ring satisfying the condition (*), U be an admissible Lie ideal of M and d be a Jordan derivation on U of M . Then d is a derivation on U of M .

Proof. By Lemma 10(iii), we have $G_\alpha(a,b)\beta[x,y]_\gamma = 0, \forall a,b,x,y \in U$ and $\alpha,\beta,\gamma \in \Gamma$.

Also, by Lemma 10(iv), $[x,y]_\gamma \beta G_\alpha(a,b) = 0, \forall a,b,x,y \in U$ and $\alpha,\beta,\gamma \in \Gamma$.

Now, $[G_\alpha(a,b),[x,y]_\gamma]_\beta = G_\alpha(a,b)\beta[x,y]_\gamma - [x,y]_\gamma \beta G_\alpha(a,b) = 0.$

Thus, $G_\alpha(a,b) \subseteq Z([U,U]_\Gamma) = Z(U) = Z(M)$, by Lemma 6 and Lemma 7 .

Therefore, $G_\alpha(a,b) \in Z(M)$. Next, we obtain

$$\begin{aligned}
2G_\alpha(a,b)\beta G_\alpha(a,b) &= G_\alpha(a,b)\beta(G_\alpha(a,b)+G_\alpha(a,b)) \\
&= G_\alpha(a,b)\beta(G_\alpha(a,b)-G_\alpha(b,a)) \\
&= G_\alpha(a,b)\beta(d(a\alpha b)-d(a)\alpha b - a\alpha d(b) - d(b\alpha a) + d(b)\alpha a \\
&\quad + b\alpha d(a)) \\
&= G_\alpha(a,b)\beta(d(a\alpha b - b\alpha a) + (b\alpha d(a) - d(a)\alpha b) + (d(b)\alpha a \\
&\quad - a\alpha d(b))) \\
&= G_\alpha(a,b)\beta(d([a,b]_\alpha) + [b,d(a)]_\alpha + [d(b),a]_\alpha) \\
&= G_\alpha(a,b)\beta d([a,b]_\alpha) + G_\alpha(a,b)\beta[b,d(a)]_\alpha \\
&\quad + G_\alpha(a,b)\beta[d(b),a]_\alpha .
\end{aligned}$$

Since and $a,b \in U$ implies that $[b,d(a)]_\alpha, [d(b),a]_\alpha \in U$.

In view of Lemma 10, we have $G_\alpha(a,b)\beta[b,d(a)]_\alpha = G_\alpha(a,b)\beta[d(b),a]_\alpha = 0.$

Therefore, we get

$$2G_\alpha(a,b)\beta G_\alpha(a,b) = G_\alpha(a,b)\beta d([a,b]_\alpha). \tag{1}$$

Now, we obtain

$$\begin{aligned}
0 &= d(G_\alpha(a,b)\beta[x,y]_\gamma + [x,y]_\gamma \beta G_\alpha(a,b)) \\
&= d(G_\alpha(a,b))\beta[x,y]_\gamma + G_\alpha(a,b)\beta d([x,y]_\gamma) + d([x,y]_\gamma)\beta G_\alpha(a,b) + [x,y]_\gamma \\
&\quad \beta d(G_\alpha(a,b)) \\
&= d(G_\alpha(a,b))\beta[x,y]_\gamma + 2G_\alpha(a,b)\beta d([x,y]_\gamma) + [x,y]_\gamma \beta d(G_\alpha(a,b)).
\end{aligned}$$

Since $G_\alpha(a,b) \in Z(M)$ implies $d([x,y]_\gamma)\beta G_\alpha(a,b) = G_\alpha(a,b)\beta d([x,y]_\gamma)$.

Therefore, we get

$$2G_\alpha(a,b)\beta d([x,y]_\gamma) = -d(G_\alpha(a,b))\beta[x,y]_\gamma - [x,y]_\gamma \beta d(G_\alpha(a,b)). \tag{2}$$

Then from (1) and (2), we have

$$\begin{aligned}
4G_\alpha(a,b)\beta G_\alpha(a,b) &= 2G_\alpha(a,b)\beta d([a,b]_\alpha) \\
&= -d(G_\alpha(a,b))\beta[a,b]_\alpha - [a,b]_\alpha \beta d(G_\alpha(a,b)).
\end{aligned}$$

Thus, we obtain

$$4G_\alpha(a,b)\beta G_\alpha(a,b)\beta G_\alpha(a,b) = -d(G_\alpha(a,b))\beta[a,b]_\alpha\beta G_\alpha(a,b) - [a,b]_\alpha\beta d(G_\alpha(a,b))\beta G_\alpha(a,b).$$

Since $[a,b]_\alpha\beta G_\alpha(a,b) = 0$ and $d(G_\alpha(a,b)) \in M$, so we have $[a,b]_\alpha\beta d(G_\alpha(a,b))\beta G_\alpha(a,b) = 0.$

Therefore, we obtain

$$4G_\alpha(a,b)\beta G_\alpha(a,b)\beta G_\alpha(a,b) = 0.$$

Since M is 2-torsion free, so we have

$$G_\alpha(a,b)\beta G_\alpha(a,b)\beta G_\alpha(a,b) = 0.$$

This shows that $G_\alpha(a,b)$ is a nilpotent element of the completely semiprime Γ-ring M, where $G_\alpha(a,b) \in Z(M)$. Since the centre of a completely semiprime Γ-ring contains no nonzero nilpotent elements, so we get $G_\alpha(a,b) = 0, \forall a,b \in M$ and $\alpha \in \Gamma$.

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