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Transcendental equation in quadratic form and its solution

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Abstract

Bisection and regular false position methods are widely used to find a root of a non-algebraic or transcendental equation. The aim of thi paper is to find a new algorithm for solving transcendental equations using three points (two end points of the interval and their mid-point to interpolate an equivalent quadratic polynomial, geometrically which represents a parabola. One of the roots of this quadratic polynomi will be an approximate root of the given transcendental equation.

Keywords: Transcendental equations, Regular false-position method, Interpolation, Quadratic equation, Taylor polynomial

Introduction

If $f(x)$ is a real and continuous function in an interval $a<x<b$, and $f(a)$ and $f(b)$ are of opposite signs, (Balagurusamy 2004) i.e. $f(a)\cdot f(b)<0$ then there is at least one real root in the interval between a and b. Let us also define another point c to be the midpoint between a and b, i.e. $c=(a+b)/2$. Using Newton's Interpolating polynomial for three points $(a, f(a))$, $(c, f(c))$, $(b, f(b))$ we can form a quadratic polynomial $g(x)$ which is an approximation of $f(x)$ on the interval $[a,b]$. A solution of $g(x)=0$ will be an approximate root of $f(x)=0$.

Formulation Using Newton's Forward Difference Formula

To find the equivalent quadratic expression of $f(x)$ we use the forward difference formula (Sastry 1999).

Table I: Forward difference table

x	y	Δ	Δ^2
a	$f(a)$	$f(c)-f(a)$	$f(a)-2f(c)+f(b)$
$c=(a+b)/2$	$f(c)$	$f(b)-f(c)$	
b	$f(b)$		

Now, from Newton's forward difference formula we get

$$g(x)=y_0+p\Delta y_0+\frac{p(p-1)}{2!}\Delta^2 y_0 \qquad (1)$$

where

$p=\frac{x-a}{c-a}$, $y_0=f(a)$, $\Delta y_0=f(c)-f(a)$,

$\Delta^2 y_0=f(a)-2f(c)+f(b)$;

and $g(x)$ is the equivalent quadratic polynomial of $f(x)$. So equation (1) takes the form

$$g(x)=f(a)+\frac{(f(c)-f(a))(x-a)}{c-a}+\frac{(f(a)-2f(c)+f(b))(x-a)(x-c)}{2(c-a)^2} \qquad (2)$$

Therefore, the above quadratic polynomial $g(x)$ is a approximation of $f(x)$ in the interval $[a,b]$. A solution of $g(x)=0$ will be an approximate root of $f(x)=0$.

The quadratic equation $g(x)=0$ can be written as,

$$(f(a)-2f(c)+f(b))x^2-[a(f(a)-4f(c)+3f(b))+b(3f(a)-4f(c)+f(b))]x+a^2f(b)+b^2f(a)+ab(f(a)-4f(c)+f(b))=0 \qquad (3)$$

Solving the quadratic equation (3) we have the solutions

$$x_1=\frac{P_1+P_2}{P_3} \qquad (4)$$

and $$x_2=\frac{P_1-P_2}{P_3} \qquad (5)$$

where

$P_1=a(f(a)-4f(c)+3f(b))+b(3f(a)-4f(c)+f(b))$

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$$P_2 = (a-b)\sqrt{\begin{matrix}(f(a))^2+16(f(c))^2+(f(b))^2\\ -2f(a)f(b)-8f(a)f(c)-8f(b)f(c)\end{matrix}}$$

$$P_3 = 4(f(a)-2f(c)+f(b))$$

Since $f(a)$ and $f(b)$ have the opposite signs, so $g(a)$ and $g(b)$ must have opposite signs. Therefore, $g(x)$ will obviously intersect the x-axis into two distinct points. Consequently, the solutions x_1 and x_2 will always be distinct and real. One of the two solutions will be the approximate root of $f(x)=0$. If $a<x_1<b$ then $x_2<a$ or $x_2>b$ and x_1 will be the approximate root of $f(x)=0$. If $a<x_2<b$ then $x_1<a$ or $x_1>b$ and x_2 will be the approximate root of $f(x)=0$. By checking these conditions in all iterations, we have to identify the formula which we have to use. Choosing the result from first iteration, replace it with one of the end points a or b on the basis of the sign of its functional value.

We can establish an algorithms whose iterations give better approximation than that of Regular False position or Bisection method for the root of $f(x)=0$, whereas the procedure is interpreted geometrically. The computation is performed by the help of the package *Mathematica*.

Algorithm

To find a solution to $f(x)=0$ given the continuous function f on the interval $[a,b]$ where $f(a)$ and $f(b)$ have opposite signs: (Burden *et al.* 2001)

INPUT endpoints a,b; tolerance TOL; maximum number of iterations m

OUTPUT approximate solution x_0 or message of failure

Step 1 Set $i=1$

Step 2 while $i\le m$ do Step 3 to Step 8

Step 3 $c=\dfrac{a+b}{2}$

Step 4 Set

$$P_1 = a(f(a)-4f(c)+3f(b))+b(3f(a)-4f(c)+f(b))$$

$$P_2 = (a-b)\sqrt{\begin{matrix}(f(a))^2+16(f(c))^2+(f(b))^2\\ -2f(a)f(b)-8f(a)f(c)-8f(b)f(c)\end{matrix}}$$

$$P_3 = 4(f(a)-2f(c)+f(b))$$

$$x_1=\frac{P_1+P_2}{P_3} \text{ and } x_2=\frac{P_1-P_2}{P_3}$$

Step 5 If $a<x_1<b$ then set $x_0=x_1$

else set $x_0=x_2$

Step 6 If $f(a)f(x_0)>0$ then set $a=x_0$

else set $b=x_0$

Step 7 If $|a-b|<TOL$ then

OUTPUT (x_0) (Procedure completed successfully)

STOP

Step 8 Set $i=i+1$

Step 9 OUTPUT (Method failed after m iterations)

(Procedure completed unsuccessfully)

STOP

Working Rule With Example

To find a root of the function (Burden *et al.* 2001)

$$f(x)=e^x+2^{-x}+\cos x-6 \quad (6)$$

over the interval using our present method we need two initial approximations a and b so that $f(a)<0$ and $f(b)>0$.

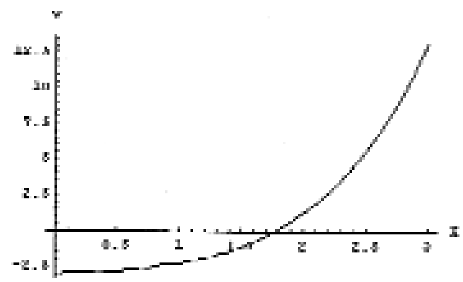


Fig. 1: Graph of $f(x)$

Since $f(1)=-2.24142$ and $f(3)=13.2205$ so, we choose $a=1$, $b=3$ and $c=\dfrac{a+b}{2}=2$. Now using equation (2) we get the equivalent quadratic equation of $f(x)$ is

$$4.26666x^2-9.33564x+2.82757=0 \quad (7)$$

Solving equation (7) we get the values $x_1=0.363151$ and $x_2=1.8249$. Since $x_2=1.8249$ lies between 1 and 3 so $x_1=0.363151$ is rejected here. As a result $x_2=1.8249$ is the first approximation. Since the functional value for $x_2=1.8249$ is positive so we replace the value $x_2=1.8249$ in stead of the value of b. Again we have to find out a quadratic equation and its solutions. Continuing this procedure we can find a root of the transcendental equation (6).

Geometric Interpretation

We know that the quadratic equation represents a parabola. In the case of our new approach, the parabola will pass through three points like $(a,f(a))$, $(c,f(c))$ and $(b,f(b))$ and cut the x axis near the zero of $f(x)$. Using the cutting point we can find again the new approximation and so on. The following figures in the Table III shows the parabolas passing through the points given in Table II.

Table II: Iterative approximations for the present method with tolerance 0.0001

No. of iteration	a	b	Approximate root p	$f(p)$
1	1.00000000000000	3.00000000000000	1.8248951407165	0.233032986984
2	1.00000000000000	1.8248951407165	1.7744830550321	-0.012750378014
3	1.8248951407165	1.7744830550321	1.7771829784546	3.33128105×10^{-6}
4	1.7744830550321	1.7771829784546	1.7771822745498	2.85290014×10^{-10}

Table III: Geometric interpretation of the iterations of the present method

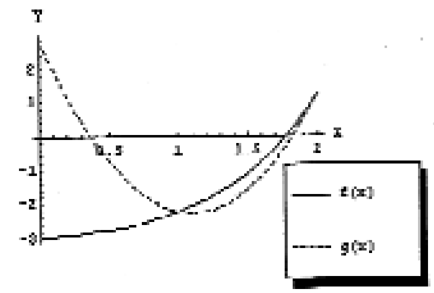
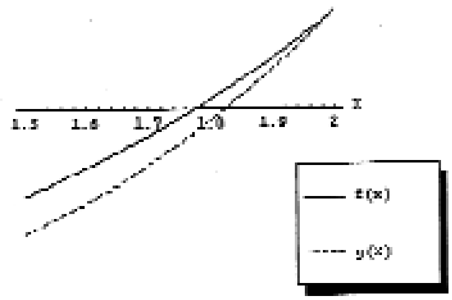
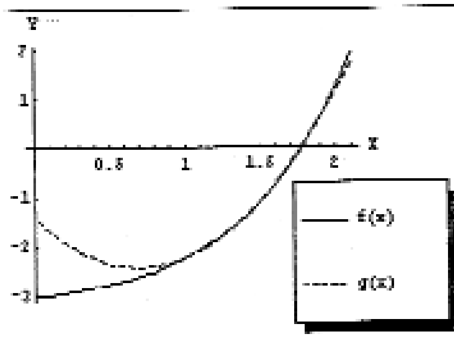
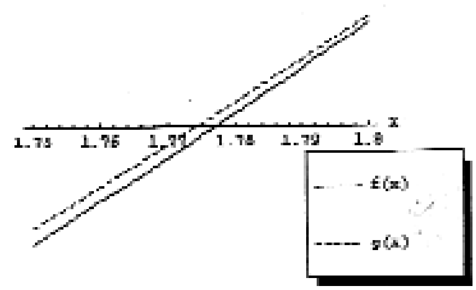
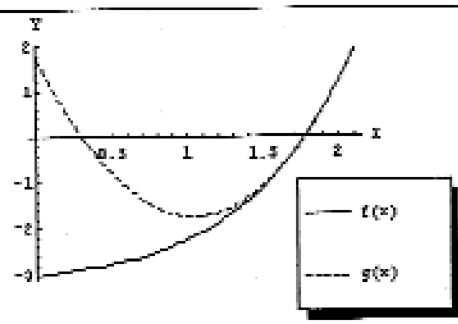
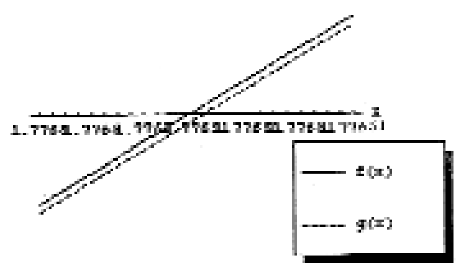
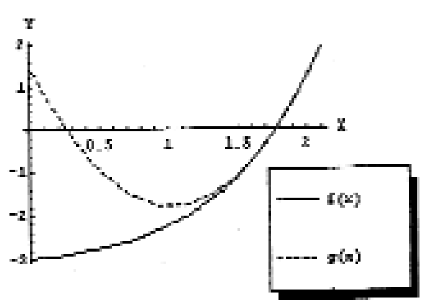
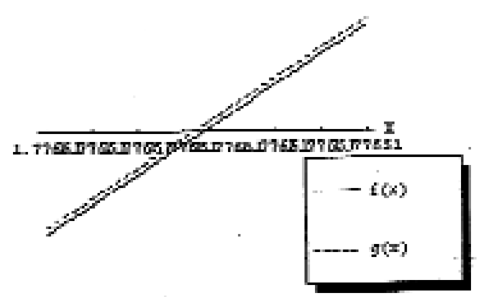
No. of iteration	Graph of $f(x)$ with quadratic approximation	
	In remote view	In close view
1		
2		
3		
4		

Table IV: Comparison of the result with the bisection method and false position method with tolerance 0.0001.

No. of iteration	Regular false-position method	Bisection method	Present method
1	1.2899264806410	2.0000000000000	1.8248951407165
2	1.4828672009263	1.5000000000000	1.7744830550321
3	1.6041635463171	1.7500000000000	1.7771829784546
4	1.6772743547939	1.8750000000000	1.7771822745498
5	1.7201273029210	1.8125000000000	
6	1.7448136944489	1.7812500000000	
7	1.7588890365717	1.7656250000000	
8	1.7668664291911	1.7734375000000	
9	1.7713722534192	1.7773437500000	
10	1.7739122998661	1.7753906250000	
11	1.7753426113893	1.7763671875000	
12	1.7761475257975	1.7768554687500	
13	1.7766003365261	1.7770996093750	
14	1.7768550184506	1.7772216796875	
15	1.7769982475894	1.7771606445313	
16	1.7770787923973	1.7771911621094	
17	1.7771240851260		
18	1.7771495540650		
19	1.7771638755616		

Comparison

Taking two end points $a=1$ and $b=3$ with tolerance 0.0001 in the three cases we observe that the false position method takes 19 iterations the bisection method takes 16 iterations and our present method takes only 4 iterations to reach the result with desired tolerance.

Conclusion

The present method provides better approximation than the regular false position and bisection method for a transcendental equation. Actually regular false-position method works with straight line and quadratic false-position method work with a parabola (a bending line) so that the root is achieved earlier. Furthermore bisection method works with mid-point and here we also use the idea of mid point. So this method is a combination of regular false-position method and bisection method.

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